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A Primer on *Rational Consequence Relations*, *Popper Functions*, and their Ranked Structures

Abstract. *Rational consequence relations* and *Popper functions* provide logics for reasoning under uncertainty, the former purely qualitative, the latter probabilistic. But few researchers seem to be aware of the close connection between these two logics. I'll show that *Popper functions* are probabilistic versions of *rational consequence relations*. I'll not assume that the reader is familiar with either logic. I present them, and explicate the relationship between them, from the ground up. I'll also present alternative axiomatizations for each logic, showing them to depend on weaker axioms than usually recognized.

Keywords: uncertain reasoning, nonmonotonic logic, conditional probability, rational consequence relations, Popper functions

1. Introduction

Rational consequence relations and *Popper functions* provide logics for reasoning under uncertainty, the former purely qualitative, the latter probabilistic. Although many researchers are already familiar with one or both of these logics, few seem to be aware of the close connection between them. It turns out that *Popper functions* are just probabilistic versions of the *rational consequence relations*. The relationship between them works like this:

1. For each *Popper function* P, the relation $\mid\!\sim_P$ defined as follows, satisfies the axioms for the *rational consequence relations*:
   $B \mid\!\sim_P A$ if and only if $P[A \mid B] = 1$.
2. For each *rational consequence relation* $\mid\!\sim$, there is a corresponding *Popper function* $P_{\mid\sim}$ such that:
   $P_{\mid\sim}[A \mid B] = 1$ if and only if $B \mid\!\sim A$;
   $P_{\mid\sim}[C \mid B] = 1$ for all C if and only if $B \mid\!\sim \neg B$;
   $P_{\mid\sim}[A \mid B] = 0$ if and only if $B \mid\!\sim \neg A$ and $B \not\mid\!\sim \neg B$;
   $0 < P_{\mid\sim}[A \mid B] < 1$ if and only if $B \not\mid\!\sim A$ and $B \not\mid\!\sim \neg A$.

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This paper describes the essential features of these two logics, features that make them particularly useful for certain kinds of projects in epistemology. I will not assume that the reader is familiar with either logic. I present each logic, and explicate the relationship between them, from the ground up. Nevertheless, even experts will find some novel ideas here. For instance, I'll present alternative axiomatizations for each logic, showing them to depend on weaker axioms than usually recognized.

Throughout this paper I'll stick to countable languages for sentential logic, and I'll treat *rational consequence relations* and *Popper functions* as residing in the meta-language (where *truth* and *logical entailment* reside). Before presenting the usual axioms, I'll first characterize these logics in terms of ranked truth-tables (on finite languages — i.e. on languages containing a finite number of sentence letters); I'll do that in section 3, where I define *truth-table consequence relations* and *truth-table conditional probability functions.* Sections 4 and 5, respectively, specify the usual axioms for *rational consequence relations* and for *Popper functions* (on both finite and countably infinite languages for sentential logic). In these sections we will also see that most of the usual axioms for each logic may be replaced by much weaker axioms. Finally, section 6 shows how to uncover the ranked structure implicit in *rational relations* and *Popper functions.* Based on this, it shows that for finite sentential languages the *truth-table consequence relations* are just the *rational consequence relations*, and the *truth-table conditional probability functions* are just the *Popper functions.* Thus, the truth-table versions of the *consequence relations* and *conditional probability functions* provide insight into the essential features of these logics.[1] In advance of all this, the next section provides an intuitive gloss of the nature of each logic.

2. The Main Idea

Rational consequence relations are nonmonotonic relations between (conjunctions of) premises and conclusions. Failure of monotonicity means that when statement B supports statement A ($B \mid\!\sim A$), the addition of a further premise C to B may undermine that support ($(B \cdot C) \not\mid\!\sim A$), and may even result in support $\neg A$ ($(B \cdot C) \mid\!\sim \neg A$). For *rational relations* the way in which monotonicity can fail is tightly constrained. Monotonicity may fail only in cases where premise B by itself supports the negation of new premise C — i.e. $B \mid\!\sim A$ and $(B \cdot C) \not\mid\!\sim A$ only when $B \mid\!\sim \neg C$. Thus, when a new premise

[1]Due to lack of space, I've placed formal statements of theorems and their proofs in an online appendix. The link is provided in the appendix section at the end of this paper.

C is added to premise B, *rational entailment* is maintained provided that B by itself doesn't support $\neg C$: if $B \mid\sim A$ and $B \not\mid\sim \neg C$, then $(B \cdot C) \mid\sim A$; this rule is called *Rational Monotonicity*, RM.

Rational entailments of form '$B \mid\sim A$' are often provided interpretative readings of the following sort: "if B, then usually A"; "if B, then normally A", "if B, then typically A". Such readings are somewhat misleading. They suggest a weaker connection between premise B and conclusion A than is implied by the logic of the *rational relations*. A more appropriate, more literal reading of '$B \mid\sim A$' is, "among *all* states of affairs that count as *credible* in the context of premise B, all those that make B true also make A true". So, a more appropriate short-hand reading is, "when B, almost certainly A". The important point is that the premise B of *rational entailment* '$B \mid\sim A$' functions in two ways: (1) it sets an epistemic context that restricts the class of epistemically possible states of affairs to those considered *credible* within that context, call these the the "B-credible states of affairs"; (2) it then draws on those B-credible states of affairs that make B true, and asserts that all of them make A true. Entailment '$B \mid\sim A$' allows that some epistemically possible states of affairs may make B true and A false, provided that in the context called for by B all such states may be "properly ignored" as too incredible to be worthy of consideration. The premise of a rational entailment cues the appropriate epistemic context, which determines the sub-class of possible states of affairs that count as "credible enough" to be relevant.[2] This epistemic context-setting is essential to the way nonmonotonicity works for *rational relations*.

Each *rational consequence relation* partitions the class of epistemically possible states of affairs into a ranked hierarchy of sub-classes. The highest ranked sub-class for a given relation $\mid\sim$ counts as the class of "most credible" possible states for $\mid\sim$. Let's call these "most credible" states the rank-1 states for $\mid\sim$. All states at lower ranks for $\mid\sim$ count as incredible with respect to (in the context of) the rank-1 states for $\mid\sim$. When B is true in some rank-1 state for $\mid\sim$, we say that B's rank is rank-1 for $\mid\sim$, and the B-credible states are just these rank-1 states. In this case B's occurrence as a premise in '$B \mid\sim A$' signals a context that draws only on the collection of rank-1 states.

When B is not true in any rank-1 states for $\mid\sim$, we assign B's rank to be the highest rank below rank-1 that contains a state that makes B true. The states collected at this rank constitute the B-credible states for $\mid\sim$; they are the states considered credible in the most prominent context where B is

[2] This conception of credible vs. incredible possibilities is closely akin to that of *properly ignored possibilities* in some contextualist accounts of knowledge (e.g. see Lewis [6]).

possibly true. In this case, B's occurrence as a premise in '$B \mid\sim A$' signals a context that draws only on the collection of B-credible states — i.e. the collection of all states at B's rank (regardless of whether they make B true). The entailment '$B \mid\sim A$' says that among these states (within this *most prominent context* where B *may* be true), each of them that makes B true also makes A true.

On this reading, the denial of a *rational entailment* of form '$B \not\mid\sim \neg C$' (it is *not* the case that $B \mid\sim \neg C$) says that "there are B-credible states of affairs where B is true and C is true." Then, these same B-credible states must also be $(B \cdot C)$-credible states (because this rank contains states that make $(B \cdot C)$ true, and it's the highest rank where B is true, so no higher rank can make $(B \cdot C)$ true). Thus, monotonicity holds in such cases as follows: "in all B-credible states of affairs where B is true, A is true" $(B \mid\sim A)$, "there are B-credible states of affairs where B is true and C is true" $(B \not\mid\sim \neg C)$, so "in all $(B \cdot C)$-credible states of affairs (which are the B-credible states of affairs) that make B true and C true make A true" $((B \cdot C) \mid\sim A)$.

Cases of nonmonotonicity always involve switching to a lower rank, which corresponds to a shift in epistemic context. Here is the common pattern: "in all B-credible states of affairs where B is true, A is true" $(B \mid\sim A)$, but "in all B-credible states of affairs where B is true, C is false" $(B \mid\sim \neg C)$; in that case the highest ranked (closest to rank-1) collection of states containing a state that makes $(B \cdot C)$ true — the $(B \cdot C)$-credible states, at $(B \cdot C)$'s rank — must be at some rank below the collection of B-credible states. There is no guarantee that those $(B \cdot C)$-credible states that make $(B \cdot C)$ true will also make A true. Indeed, '$(B \cdot C) \mid\sim A$' will fail to hold just in case some $(B \cdot C)$-credible state makes $(B \cdot C)$ true and A false.

The axioms for *rational consequence relations* do not explicitly refer to these ranked hierarchies of classes of states. Rather, we may derive them from the axioms. We'll see how that works in section 6. However, before we get to the axioms for *rational relations*, I'll first characterize the ranked hierarchy of classes of states in another way — in terms of ranked truth-tables, where the ranked hierarchy of classes of states is represented by a ranked hierarchy of classes of truth-table lines.

Here is a typical example of a *rational relation.* In this case the relation $\mid\sim$ associates with the statement 'the animal is a bird' a context that excludes as *too incredible* those states of affairs where the animal in question may belong to a non-flying bird species. However, $\mid\sim$ associates with the conjunctive statements "the animal is a bird and an ostrich" and "the animal is a bird and a penguin" alternative contexts (at lower levels) containing possible states where the animal in question may belong to a non-flying species.

> My brother, who lives with his family in New Hampshire, tells me he may buy his daughter a pet for her tenth birthday, perhaps some kind of bird. Here are some sensible *rational entailments* relevant to this situation.
>
> Abbreviations for statements: '$Bird$' = "the animal (the new pet) is a bird"; '$Flies$' = "the animal is a member of a species that can fly"; '$African$' = "the animal is a member of a species native to Africa"; '$Antarctic$' = "the animal is a member of a species native to the Antarctic"; '$Ostrich$' = "the animal is a species of ostrich"; '$Penguin$' = "the animal is a species of penguin":
>
> (1) $Bird \mid\!\sim Flies$; (2) $Bird \mid\!\not\sim \neg African$; (3) $Bird \cdot African \mid\!\sim Flies$;
> (4) $Bird \cdot Ostrich \mid\!\sim \neg Flies$; (5) $Bird \cdot African \cdot Ostrich \mid\!\sim \neg Flies$;
> (6) $Bird \cdot Antarctic \mid\!\not\sim Flies$; (7) $Bird \cdot Antarctic \mid\!\not\sim \neg Flies$;
> (8) $Bird \cdot Antarctic \cdot Penguin \mid\!\sim \neg Flies$;
> (9) $Bird \cdot Antarctic \cdot \neg Penguin \mid\!\sim Flies$.
>
> The rules for *rational consequence relations* permit the derivation of (3) from (1) and (2), of (10) '$Bird \mid\!\sim \neg Ostrich$' from (1) and (4), and of (11) '$Bird \mid\!\sim \neg Antarctic$' from (1) and (6).

Here, '$Bird$' must trigger the same context as '$Bird \cdot Africian$'. '$Bird \cdot Ostrich$' and '$Bird \cdot Antarctic$' must trigger new contexts at ranks below that of 'Bird'. However, '$Bird \cdot Antarctic$', '$Bird \cdot Antarctic \cdot Penguin$', and '$Bird \cdot Antarctic \cdot \neg Penguin$' may all share the same lower rank.

In many cases it seems natural to extend a *rational relation* to a conditional probability function. For example, given that '$Bird \cdot Antarctic \mid\!\not\sim Flies$' and '$Bird \cdot Antarctic \mid\!\not\sim \neg Flies$', perhaps the above *rational relation* $\mid\!\sim$ may reasonably be extended to conditional probability function P where: $P[Flies \mid Bird \cdot Antarctic] = .1$ and $P[\neg Flies \mid Bird \cdot Antarctic] = .9$. This can indeed be done. Each *rational consequence relation* $\mid\!\sim$ can be extended to a conditional probability function that assigns $P[A \mid B] = 1$ whenever $B \mid\!\sim A$, and assigns a probability value strictly between 0 and 1 to $P[A \mid B]$ whenever $B \mid\!\not\sim A$ and $B \mid\!\not\sim \neg A$. In each such extension, the corresponding conditional probability function P turns out to be a *Popper function*.

Popper functions extend the logic of classical conditional probability by making non-trivial use of statements that have probability 0.[3] For classical unconditional probability functions p, conditional probability is defined as

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[3]Named after Karl Popper, who developed them in an appendix to the 1959 edition of *The Logic of Scientific Discovery*, [8]. Makinson [7] for a thorough comparison of approached to conditional probability functions.

by the ratio formula: whenever $p[B] > 0$, $p[A \mid B] = p[A \cdot B]/p[B]$; whenever $p[B] = 0$, $p[A \mid B]$ is left undefined. A minor modification of this usual approach is to require $p[A \mid B] = 1$ whenever $p[B] = 0$. This permits conditional probabilities to remain defined when the condition has 0 probability. Specified this way, each classical conditional probability function is a very simple kind of *Popper function.* More generally, each *Popper functions* turns out to be a conditional probability function that consist of a ranked hierarchy of classical probability functions, where conditionalization on a sentence that has 0 probability at one rank brings about a shift to a different classical probability function that lies at a lower rank in the hierarchy. That is, for *Popper function* P, the function $P[\ldots \mid B]$ (holding statement B fixed) behaves as a classical probability function; and for $P[C \mid B] > 0$, $P[A \mid B \cdot C] = P[A \cdot C \mid B]/P[C \mid B]$, matching the way probabilistic conditionalization is defined classically. However, when $P[C \mid B] = 0$, the function $P[\ldots \mid B \cdot C]$ gotten by holding $(B \cdot C)$ fixed behaves as a distinct classical probability function, one that resides at a lower rank in a hierarchy of classical probability functions represented by P.

The existence of this ranked structure is not explicitly referred to by the axioms for *Popper functions*, but can be derived from them. This ranked structure is precisely of the kind derivable from the axioms for *rational consequence relations.* P generates a ranked hierarchy of classes of possible states of affairs together with an associated rank-specific weighting function on the sentences of each rank. The rank of sentence B is the highest rank where some state of affairs makes B true and where the associated rank-specific function p assigns B a non-0 weight. So, when B occurs as the "premise condition" for a conditional probability expression of form $P[A \mid B]$, it cues the context for B's rank and associated rank-specific probability function p, and assigns $P[A \mid B]$ the value $p[A \cdot B]/p[B]$. Thus, *Popper functions* are just *rational consequence relations* supplemented with a weighting function for sentences at each rank. We'll see how this works in sections 3 and 6.

A *Popper function* may be used to represent a hierarchy of classical Bayesian belief functions appropriate to distinct epistemic contexts. Consider, for example, cases where it makes good sense to represent a hierarchy of increasingly more skeptical contexts. In the least skeptical (highest level) context, the *Popperian credibility function* treats all skeptical hypotheses as too incredible to count among the "real possibilities", and assigns them 0 credence. But conditionalization on evidence for a skeptical claim shifts to a more open-minded context (at a lower level of the hierarchy), where a broader range of hypotheses are included among the class of "credible possibilities". But even within this more open-mined context, a range of even more highly skeptical claims may count as "too incredible", and so be given 0 credence. Nevertheless, the acquisition of evidence for these claims can, via conditionalization, bring the *Popperian credibility function* to shift to a yet lower level context where even these claims may count among the credible options, and receive positive credence.

More generally, suppose that each of several distinct but related epistemic contexts are best represented by a different classical probability function. Rather than deal with this situation piecemeal, one can represent all of these probability functions together with their associated contexts in terms of a single *Popper function* together with appropriate context-indicating (context-shifting) statements. Conditionalization on an appropriate context-indicating statement induces the *Popper function* to bring the appropriate classical probability function online. The *Popper function* implements this by arranging the collection of classical probability functions into a hierarchy. It treats the classical probability function at the top of the hierarchy as the most appropriate function to use. Conditionalization on a statement C that has 0 probability at this level brings online the highest level classical probability function in the hierarchy that provides statement C a non-0 probability. This lower-level probability function remains online until some statement (or conjunction of statements) D that has probability 0 at that level is conditionalized on. This brings online an even lower level probability function, the highest level probability function in the hierarchy that provides non-0 probability to statement D. The logic by which *Popper functions* implement this context-switching turns out to be precisely the logic of the *rational consequence relations.*

Thus, for *Popper functions* probability 0 need not mean "is impossible" or "is certainly false", and probability 1 need not mean "is certainly true". Rather, for *Popper functions* probability 1 may mean "is almost certain" or "is provisionally certain" — i.e. "is almost surely true, given the options considered to be real possibilities in the present context."

The remaining sections are strictly formal. They present axioms for the *rational consequence relation* (section 4) and for the *Popper functions* (section 5), explicate their ranked structures (section 6), and explicate the formal features I've attributed to them thus far. I'll leave their proofs to an appendix (available online, due to lack of space here). Throughout I'll stick to standard languages for sentential logic. However, many of the results described here can be extended to languages for predicate logic.[4]

[4]See [1], [2] for treatments of *Popper functions* and *rational consequence relations* on languages for predicate logic.

3. Ranked Truth-Tables

The essential features of the *rational consequence relations* and the *Popper functions*, and the relationships between them, are most easily comprehended in terms of the notion of *ranked truth-tables* on finite languages for sentential logic (i.e. standard sentential languages containing only finitely many sentence letters). In this section I'll spell out the notion of a *ranked truth-table* and define the notion of a *truth-table consequence relation.* For finite languages these consequence relations turn out to be coextensive with the *rational consequence relations* (as explicated in section 6 and proved in the appendix). Then I'll extend *ranked truth-tables* to *probabilistic ranked truth-tables* and define the notion of a *truth-table conditional probability function.* For finite languages these functions turn out to be coextensive with the *Popper functions* (as explicated in section 6 and proved in the appendix). Thus, the approach through ranked truth-tables provides a simple, easy to understand characterization of the essential natures of the *rational consequence relations*, the *Popper functions*, and the relationship between them.

Let L be any finite language (or finite part of a countably infinite language) for sentential logic. Suppose L contains n sentence letters. Here is how to construct a ranked truth-table.

- First construct a truth-table of the usual sort for the sentence letters of L (consisting of 2^n truth-table lines where each line provides a unique truth-value assignment to the sentence letters).
- At the top left corner of the table, immediately to the left of the first sentence letter, write 'rank' and construct a new column below it.
- Under the 'rank' column, beside one or more truth-table lines, write '1'; these are the rank-1 truth-table lines (the "highest ranked" lines).
- Perhaps all of the lines are marked '1'; but if not, then you may mark some or all of the remaining lines with a '2'; these are the rank-2 lines (the second highest ranked lines).
- If some lines have been marked '2', and if any lines remain unmarked, then you may mark some or all of them with a '3'.
- .... Continue in this way, marking as many ranks as you wish, up through the entire 2^n lines (for n sentence letters), if you wish. But you may leave some of the lines un-numbered — some lines may not possess a finite rank.

- Finally, if any un-numbered lines remain, mark them 'ω'; these rank-ω lines, if any, will be on a par with contradictions — they will count as "absolutely impossible" in a way that will be explained in a moment.

The rank-1 lines are the highest ranked lines; rank-2 lines are the next highest ranked; ...; rank-ω lines are at the lowest ranked. Given such a ranked truth-table T for language L, we next assign ranks to sentences of L:

- For each sentence B of L, the rank of B for truth-table T is the highest rank (lowest rank number) that T assigns to any line on which B is true;
- if B is a contradiction, then the rank of B is ω.
- The rank-B lines of T are all lines having the same rank as B (regardless of whether they are lines that make B true).
- The rank-B sentences of T are those having the same rank as B.

Associate with each such ranked truth-table T the corresponding T *truth-table consequence relation*, defined as follow.

Definition: Truth-table Consequence Relation: The *truth-table consequence relation* generated by ranked truth-table T is the relation $\mid\!\sim_T$ such that, for all sentences A and B in T's language, $B \mid\!\sim_T A$ *if and only if*

1. the rank of B is ω ; or
2. the rank of B is not ω, and every rank-B line that makes B true also makes A true.

Notice that when the rank of B is ω, '$B \mid\!\sim_T A$' holds for all sentences A.

Every *truth-table consequence relation* satisfies the axioms for *rational consequence relations* specified in section 4. Furthermore, each *rational consequence relation* on a finite language can be generated as a *truth-table consequence relation* for some ranked truth-table (see section 6 and the appendix). So, the workings of the *truth-table consequence relations* captures the essence of the *rational consequence relations.*

Notice that whenever $B \not\mid\!\sim_T A$ and $B \not\mid\!\sim_T \neg A$, some rank-$B$ lines of T that make B true must make A false (otherwise we would have $B \mid\!\sim_T A$) and some rank-B lines that make B true must make A true (otherwise we would have $B \mid\!\sim_T \neg A$). One can well imagine that for a specific application some truth-table lines within the same rank may count as more probable than others. This suggests extending the notion of a ranked truth-table T to that of a probabilistic ranked truth-table, T_P. Here is how to do that.

- First construct a ranked truth-table T as described above.
- Add a new column to T; place this column between the column labeled 'rank' and the column for the first sentence letter, and label it 'weight'.
- In this new column, next to each line of finite rank (each line not labeled ω), write an expression for a positive real number; leave the "weight" entry blank for the rank ω lines.

It would be natural to assign weights that sum to 1 within each rank. The present construction permits this, but doesn't require it. I'll say more about that in a moment.

Associate with each such probabilistic ranked truth-table T_P a corresponding truth-table probability function P, defined as follows.

Definition: Truth-table Probability Function: The *truth-table conditional probability function* generated by probabilistic ranked truth-table T_P is the function P such that for all sentences A and B of T_P's language, $P[A \mid B] = r$ *if and only if*

1. the rank of B is ω and $r = 1$; or
2. $r =$ the sum of the weights of the rank-B lines that make $(A \cdot B)$ true divided by the sum of the weights of the rank-B lines that make B true.

Clause (i) implies that whenever the rank of B is ω , $P[A \mid B] = 1$ for all A. Furthermore, applying the definition of *truth-table consequence relation* to the ranks of a *probabilistic ranked truth-table* T_P generates a *truth-table consequence relation* $|\!\sim_T$ such that $P[A \mid B] = 1$ for T_P just in case $B \mid\!\sim_T A$.

When assessing the conditional probability values for function P, clause (ii) automatically normalizes the weights within each rank. That is, multiplication of each weight within a rank by a constant positive factor c will result in precisely the same probability values, since factor c will get "divided out" when the probability values are computed from the weights. So, although we can define the probabilistic truth-tables so as to require that the weights of lines be probabilities (positive real numbers that sum to one) within each rank, we may just as well permit the weights of lines to be positive real numbers of any size. From the non-probabilistic weights we can always generate a corresponding probabilistic weight for each truth-table line — just divide each line's weight by the sum of all the weights of lines within its own rank.

A *probabilistic ranked truth-table* contains a classical probability function at each finite rank. To fully specify the classical probability function p_q at rank-q of the *probabilistic ranked truth-table* T_P:

at each rank q, for each sentence B (regardless of whether it is a rank-q sentence), define $p_q[B]$ = the sum of weights of rank-q lines that make B true divided by the sum of the weights of all rank-q lines.

It's easy to see that for the ranked truth-table T_P, the classical probability function at each rank is related to the truth-table probability function P as follows: $P[A \mid B] = p_q[A \cdot B]/p_q[B]$ whenever the rank of B is q.

Clearly, for each probabilistic ranked truth-table T_P generated by a ranked truth-table T, the truth-table probability function P is related to the truth-table consequence relation $\mid\!\sim_T$ in the following way:

$P[A \mid B] = 1$ just in case $B \mid\!\sim_T A$;

$P[C \mid B] = 1$ for all C if and only if $B \mid\!\sim_T \neg B$;

$P[A \mid B] = 0$ if and only if $B \mid\!\sim_T \neg A$ and $B \not\mid\!\sim_T \neg B$;

$0 < P[A \mid B] < 1$ if and only if $B \not\mid\!\sim_T A$ and $B \not\mid\!\sim_T \neg A$.

Every *truth-table conditional probability function* satisfies the axioms for *Popper functions* specified in section 5. Furthermore, each *Popper function* on a finite language can be generated as a *truth-table conditional probability function* for some ranked probabilistic truth-table (see section 6 and the appendix). So, the workings of the *truth-table probability functions* captures the essence of the *Popper functions.*

4. Axioms for *Rational Consequence Relations*

The *rational consequence relations* are usually characterized by first specifying the axioms for the *preferential consequence relation*, then adding the *rational monotonicity axiom* RM. This axiom is really quite strong. We'll see that in its presence we can considerably weaken the other usual axioms. Here are the usual axioms for the *preferential consequence relation.*

Axioms for the *Preferential Consequence Relations*: A *Preferential Consequence Relation* on a language L for sentential logic is any relation $\mid\!\sim$ between pairs of sentences of L that satisfies the following axioms:

0. For some E, F, $E \not\mid\!\sim F$ (NT: Non-Triviality)
1. $A \mid\!\sim A$ (R: Reflexivity)
2. If $B \models C$, $C \models B$, $B \mid\!\sim A$, then $C \mid\!\sim A$ (LCE: Left Classical Equiv.)
3. If $C \mid\!\sim B$, $B \models A$, then $C \mid\!\sim A$ (RW: Right Weakening)

4. If $C \mid\sim A$, $B \mid\sim A$, then $(C \vee B) \mid\sim A$ (OR: left disjunction)
5. If $B \mid\sim A$, $B \mid\sim C$, then $(B \cdot C) \mid\sim A$ (CM: Cautious Monotonicity)
6. If $C \mid\sim B$, $C \mid\sim A$, then $C \mid\sim (B \cdot A)$ (AND: right conjunction)

I've added axiom 0 to the usual axioms to eliminate the trivial relation that has each sentence *preferentially entail* every sentence. The usual axioms for the *rational consequence relations* are those for the *preferential consequence relations* plus the Rational Monotonicity rule, RM.

Usual Axioms for the *Rational Consequence Relations*: A *Rational Consequence Relation* on a language for sentential logic is any *Preferential Consequence Relation* that satisfies the following additional axiom:

7. If $B \mid\sim A$, $B \not\mid\sim \neg C$, then $(B \cdot C) \mid\sim A$ (RM: Rational Monotony)

The monotonicity rule CM for the *preferential relations* has a stronger antecedent condition than does RM, which makes CM a weaker monotonicity rule than RM. In the presence of the other axioms it is easy to derive CM from RM. Indeed, RM turns out to be quite a strong rule, strong enough that in its presence we can get by with much weaker axioms than those supplied by the *preferential relations*. Compare the following axioms for *rational relations* with axioms 0-7.

Weak Axioms for the *Rational Consequence Relations*: A *weak Rational Consequence Relation* on a language for sentential logic is any relation $\mid\sim$ between pairs of its sentences that satisfies the following axioms:

0. for some E, F, $E \not\mid\sim F$ (NT)
1. $A \mid\sim A$ (R)
2. If $B \models C$, $C \models B$, $B \mid\sim A$, then $C \mid\sim A$ (LCE)
3. If $C \mid\sim B$, $B \models A$, then $C \mid\sim A$ (RW)
4. If $(B \cdot C) \mid\sim A$, $(B \cdot \neg C) \mid\sim A$, then $B \mid\sim A$ (WOR:Weak OR)
5. If $B \mid\sim (C \cdot A)$, then $(B \cdot C) \mid\sim A$ (VCM: Very Cautious Monotony)

6.1 If $(C \cdot B) \mid\sim A$, then $(C \cdot B) \mid\sim (B \cdot A)$ (VWAND: Very Weak AND)

6.2 If $B \mid\sim A$, $B \mid\sim \neg A$, then $B \mid\sim C$ (CNTRA: ConTRAdiction)

7. If $B \mid\sim A$, $B \not\mid\sim \neg C$, then $(B \cdot C) \mid\sim A$ (RM)

This Weak Axiomatization keeps Usual Axioms 0-3 and 7, but significantly weakens 4-6. All the Weak Axioms are easily derivable from the

Usual Axioms. Conversely, from the Weak Axioms, AND, CM, and OR are derivable (see appendix). Thus, the Weak Axioms characterize precisely the same consequence relations as the Usual Axioms.[5]

In developing what we now call the *Popper functions*, Karl Popper was not simply trying to axiomatize a generalized version of conditional probability functions. Rather, a primary objective was to provide a *logic of conditional probability* that is completely autonomous from classical deductive logic. So, Popper developed an *autonomous axiomatization* for conditional probabilities, where the axioms do not rely on classical deductive logic in any way. (Section 5 will present such axiom for *Popper functions.*)

The *rational consequence relations* can be provided this kind of autonomous axiomatization. Among the Weak Axioms, only axioms 2 and 3 draw on the classical notion of logical entailment. We can make do with weaker axioms that don't presuppose classical deductive logic in any way. The logic of the *rational consequence relations* can be taken as basic, and the classical logical entailment relation can then be shown to fall out as a special *rational relation*, the one that consists of precisely those *rational entailments* $B \mid\!\sim A$ shared by every *rational consequence relations.*

Autonomous Axioms for *Rational Consequence Relations*: An *autonomous Rational Consequence Relation* on a language for sentential logic is any relation $\mid\!\sim$ between its sentences that satisfies the following axioms: [6]

0. For some E, F, $E \not\mid\!\sim F$ (NT)
1. $A \mid\!\sim A$ (R)
2. If $(C \cdot B) \mid\!\sim A$, then $(B \cdot C) \mid\!\sim A$ (LC: Left Commutivity)

3.1 If $C \mid\!\sim (B \cdot A)$, then $C \mid\!\sim B$ (SMP-L: Simplification-Left)

3.2 If $C \mid\!\sim (B \cdot A)$, then $C \mid\!\sim A$ (SMP-R: Simplification-Right)

3.3 If $B \mid\!\sim \neg\neg A$, then $B \mid\!\sim A$ (DN: Double Negation)

3.4 If $C \mid\!\sim (\neg(B \cdot A) \cdot B)$, then $C \mid\!\sim \neg A$ (SYL: Syllogism)

4. If $(B \cdot C) \mid\!\sim A$, $(B \cdot \neg C) \mid\!\sim A$, then $B \mid\!\sim A$ (WOR)

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[5]The usual semantics for *preferential consequence relations* draws on *ranked models* of a specific kind, and the above axioms are shown to be sound and complete for this semantics (see [4]). The usual semantics for *rational consequence relations* draws on ranked models that satisfy an additional "smoothness" condition (see [5]).

[6]These axioms rely only on negation and conjunction. The other logical connectives are treated as abbreviations, in the usual way: '$(A \vee B)$' abbreviates '$\neg(\neg A \cdot \neg B)$', '$(A \supset B)$' abbreviates '$\neg(A \cdot \neg B)$', '$(A \equiv B)$' abbreviates '$(\neg(A \cdot \neg B) \cdot \neg(\neg A \cdot B))$'.

5. If $B \mid\!\sim (C \cdot A)$, then $(B \cdot C) \mid\!\sim A$ (VCM)

6.1 If $(C \cdot B) \mid\!\sim A$, then $(C \cdot B) \mid\!\sim (B \cdot A)$ (VWAND)

6.2 If $B \mid\!\sim A$, $B \mid\!\sim \neg A$, then $B \mid\!\sim C$ (CNTRA)

7. If $B \mid\!\sim A$, $B \not\mid\!\sim \neg C$, then $(B \cdot C) \mid\!\sim A$ (RM)

None of these axioms draws on classical deductive logic in any way (nor is substitutivity of logically equivalent sentences supposed here). Each of the 3.x axioms follows directly from Weak axiom 3, and autonomous axiom 2 follows directly from Weak axiom 2. Weak axiom 6 (AND) implies 6.1 and 6.2, but these two replacements are significantly weaker.[7] So, each axiom for the *Autonomous rational consequence relations* is derivable from the axioms for *Weak rational consequence relations*. We can also establish the converse — the Weak Axioms are derivable from the Autonomous Axioms (see appendix). Thus, the *Autonomous rational consequence relations* are identical to the usual *rational consequence relations*.

5. Axioms for *Popper Functions*

The axioms for *Popper functions* take conditional probability as basic, not defined in terms of unconditional probability. I'll first specify a commonly used set of axioms for them. Then I'll present a very weak-looking *autonomous axiom set*, which doesn't at all draw on classical deductive logic.

Usual Axioms for *Popper Functions*: A *Popper function* on a language for sentential logic is any function P from pairs of its sentences to real numbers that satisfies the following axioms:

0. For some E, F, G, H, $P[E \mid F] \neq P[G \mid H]$
1. $P[A \mid B] \geq 0$
2. If $C \models B$, $B \models C$, then $P[A \mid B] = P[A \mid C]$
3. If $B \models A$, then $P[A \mid B] = 1$
4. $C \models \neg(B \cdot A)$, then $P[(A \vee B) \mid C] = P[A \mid C] + P[B \mid C]$ or $P[D \mid C] = 1$ for all D
5. $P[(A \cdot B) \mid C] = P[A \mid (B \cdot C)] \times P[B \mid C]$

[7] E.g., as measured by the notion of *probabilistic consequence* (see[3]). Substituting $p[X \mid Y] \geq t$ in place of $Y \mid\!\sim X$ throughout axioms 0-6.1, each axiom holds for every probability function p and threshold value $t > 0$, and 6.2 holds for every $t > 1/2$. However, 7 (RM) holds only for threshold $t = 1$, as does previous axiom 6 (AND).

These axioms do not explicitly require $P[A \mid B] \leq 1$. That's derivable.[8]

We can axiomatize the *Popper functions* in a way that does not rely on classical deductive logic. These axioms won't even assume that the values of the functions they characterize lie between 0 and 1.[9]

Autonomous Axioms for the *Popper Functions*: An *autonomous Popper function* on a language for sentential logic is any function P from pairs of its sentences to real numbers that satisfies the following axioms:

0. For some E, F, G, H, $P[E \mid F] \neq P[G \mid H]$
1. $P[A \mid A] \geq P[B \mid B]$
2. $P[A \mid (B \cdot C)] \geq P[A \mid (C \cdot B)]$
3. $P[A \mid C] \geq P[(A \cdot B) \mid C]$
4. $P[A \mid B] + P[\neg A \mid B] = P[B \mid B]$ or $P[D \mid B] = P[B \mid B]$ for all D
5. $P[(A \cdot B) \mid C] = P[A \mid (B \cdot C)] \times P[B \mid C]$

The Autonomous Axioms are clearly derivable from the Usual Axioms. We can also establish the converse (with a lot of effort — see the appendix).

Each *truth-table probability function* is a *Popper function.* This is proved by checking that *truth-table probability functions* satisfy the axioms for *Popper functions.* The converse, that each *Popper function* on a finite language is a *truth-table probability function*, derives from showing that each *Popper function* gives rise to a ranked hierarchy of non-overlapping classes of sentences, and using that ranked hierarchy to produce a *probabilistic ranked truth-table* whose *truth-table probability function* agrees with the associated *Popper function.* The next section shows how to derive the appropriate ranked structures from the axioms for *Popper functions.*

6. How Ranked Structures Derive from the Axioms

Rational consequence relations and *Popper functions* share a common ranked structure that lies at the heart of how non-monotonicity works in these logics. In this section we'll first see how the ranked structures of *rational consequence relations* may be derived from their axioms. The ranked structures of *Popper functions* derives from their axioms in the same way, since the probability 1 parts of *Popper functions* are *rational consequence relations*,

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[8]Suppose $P[A \mid B] > 1$. Then, from axioms 2, 4, and 1 we derive a contradiction: $1 = P[A \vee \neg A \mid B] = P[A \mid B] + P[\neg A \mid B]$, so $0 > 1 - P[A \mid B] = P[\neg A \mid B] \geq 0$.

[9]Popper first provided this kind of axiomatization in an appendix to [8].

and so generate the same ranked structure. I'll only summarize the important facts about how the rankings arise, and their implications for the relationship between *rational consequence relations* and *Popper functions*. I leave the details, including proofs of these claims, to the appendix.

For each *rational consequence relation* $\mid\!\sim$, we define an ordering relation $\geq_{\mid\sim}$ on the sentences of its language. We then show this relation to be a *total preorder* on sentences of the language of $\mid\!\sim$ — i.e. $\geq_{\mid\sim}$ is a *complete, transitive relation* on sentences of the language of $\mid\!\sim$. This relation provides a ranking of the sentences of $\mid\!\sim$, where A and B have the same rank just when $A \geq_{\mid\sim} B$ and $B \geq_{\mid\sim} A$.

Definition: The Rank-Orderings of Sentences Imposed by *Rational Consequence Relations*: For each *rational consequence relation* $\mid\!\sim$, define the relation $\geq_{\mid\sim}$ on sentences of its language as follow:

'$A \geq_{\mid\sim} B$' abbreviates 'either $A\vee B \not\mid\!\sim \neg A$ or $A\vee B \mid\!\sim \neg B$'; read '$A \geq_{\mid\sim} B$' as "the rank of A is at least as high as the rank of B for $\mid\!\sim$."

1. '$A \approx_{\mid\sim} B$' abbreviates '$A \geq_{\mid\sim} B$ and $B \geq_{\mid\sim} A$'; read '$A \approx_{\mid\sim} B$' as "A and B have the same rank for $\mid\!\sim$."
2. '$A >_{\mid\sim} B$' abbreviates '$A \geq_{\mid\sim} B$ and *not* $B \geq_{\mid\sim} A$'; read '$A >_{\mid\sim} B$' as "the rank of A is higher than the rank of B for $\mid\!\sim$."
3. By definition, B has rank-ω for $\mid\!\sim$ just when $B \mid\!\sim \neg B$.
4. By definition, B has rank-1 for $\mid\!\sim$ just when $(C \vee \neg C) \not\mid\!\sim \neg B$.

Rank-1 is the rank of tautologies: each tautology D has rank-1, and $D \geq_{\mid\sim} E$ for every sentence E. Furthermore, if B has rank-1, then for every tautology D, $B \approx_{\mid\sim} D$.[10]

Rank-ω is the rank of contradictions: each contradiction D has rank-ω, and $E \geq_{\mid\sim} D$ for every sentence E. Furthermore, when B has rank-ω: (1) $B \mid\!\sim E$ for all E; and (2) for every contradiction D, $B \approx_{\mid\sim} D$.[11]

Whenever $(A \vee B) \mid\!\sim \neg A$ and $(A \vee B) \mid\!\sim \neg B$, both A and B have rank-ω.[12] So, the definition of '$A \geq_{\mid\sim} B$' yields the following characterization:

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[10]Suppose D is some tautology. Then $(C \vee \neg C) \not\mid\!\sim \neg D$, so D has rank-1. Also, $D \vee E \not\mid\!\sim \neg D$ (since $D \vee E$ is logically equivalent to D), so $D \geq_{\mid\sim} E$. When B is rank-1, $B \geq_{\mid\sim} D$ (since $B \vee D$ is logically equivalent to $(C \vee \neg C)$, so $B \vee D \not\mid\!\sim \neg B$).

[11]Suppose D is some contradiction. Then $D \mid\!\sim \neg D$, so D has rank-ω. Also, $E \vee D \mid\!\sim \neg D$ (since $\neg D$ is a tautology), so $E \geq_{\mid\sim} D$. When B has rank-ω, $D \geq_{\mid\sim} B$ (since $B \mid\!\sim \neg B$ and $D \vee B$ is logically equivalent to B, so $D \vee B \mid\!\sim \neg B$).

[12]$(A \vee B) \mid\!\sim (\neg A \cdot \neg B)$ (AND), $(A \vee B) \mid\!\sim ((A \vee B) \cdot (\neg A \cdot \neg B))$ (R, AND), $(A \vee B) \mid\!\sim (A \cdot \neg A)$ (RW), $((A \vee B) \cdot A) \mid\!\sim \neg A$ (VCM), $A \mid\!\sim \neg A$ (LCE). Similarly, $B \mid\!\sim \neg B$.

$A \geq_{\mid\!\sim} B$ if and only if $A \vee B \mid\!\not\sim \neg A$ *or* both A and B have rank-ω.

Thus, whenever either A or B does not have rank-ω:

1. $A \geq_{\mid\!\sim} B$ just when $A \vee B \mid\!\not\sim \neg A$;
2. $A >_{\mid\!\sim} B$ just when $A \vee B \mid\!\sim \neg B$;
3. $A \approx_{\mid\!\sim} B$ just when both $A \vee B \mid\!\not\sim \neg A$ and $A \vee B \mid\!\not\sim \neg B$.

The relation $\geq_{\mid\!\sim}$ is *transitive* and *complete* (proved in the appendix).[13] So, the associated strict order relation $>_{\mid\!\sim}$ and equivalence relation $\approx_{\mid\!\sim}$ together impose a (complete, transitive) ranking on sentences of $\mid\!\sim$.

We can use the ranked structure for $\mid\!\sim$ to establish the following result: For each finite sub-language of the language for $\mid\!\sim$ that contains both A and B,'$B \mid\!\sim A$' holds *just in case*, every state-description S that has the same rank as B and logically entails B also logically entails A (see appendix).[14]

This result holds regardless of whether the full language for $\mid\!\sim$ is finite or countably infinite. Given the correspondence between state-descriptions and truth-table lines, it follows that each *rational consequence relation* $\mid\!\sim$ defined on a finite language is a *truth-table consequence relation.* Furthermore, each *truth-table consequence relation* can be shown to satisfy the axioms for *rational consequence relations.* So, on finite languages the *truth-table consequence relations* just are the *rational consequence relations.*

For any given *rational relation* $\mid\!\sim$, we can supply it a weighting function w_q on sentences at each rank q above ω. From these weightings we can then define a conditional probability function $P_{\mid\!\sim}$ such that:

$P_{\mid\!\sim}[A \mid B] = 1$ when the rank of B is ω; and

$P_{\mid\!\sim}[A \mid B] = w_q[A \cdot B]/w_q[B]$ when the rank of B is q (q not ω).

Thus, each *rational consequence relation* $\mid\!\sim$ can be extended to a *Popper function* for which $\mid\!\sim$ is the probability 1 part. (The appendix shows how to provide weighting functions on sentences at each rank.)

The relation $\mid\!\sim_P$ defined as the probability 1 part of *Popper function* P is a *rational consequence relation.* So, all of the above results about rankings applies to the relation $\geq_P$ generated by each *Popper function* P. Each *Popper function* is representable by a ranked hierarchy of sentences together with a weighting function w_q for each rank q above ω. Formally, given a *Popper function* P, we can identify a collection of rank-weight pairs $< q, w_q >$ such that:

[13] "*Completeness*" means that for any two sentences A, B, either $A \geq_{\mid\!\sim} B$ or $B \geq_{\mid\!\sim} A$.

[14] A *state-description* (on a finite language) is a conjunction of sentence letters and their negations that contains each sentence letter or its negation (but not both).

$P[A \mid B] = 1$ when the rank of B is ω;

$P[A \mid B] = w_q[A \cdot B]/w_q[B]$ when the rank of B is q (q not ω).

For finite languages, the weighting function for each rank q, w_q, assigns weights to state-descriptions at that rank. This effectively assigns a rank and weight to the true-table lines that make the associated state-descriptions true. The resulting *probabilistic ranked truth-table* yields the truth-table probability function P in the way specified above. Thus, each *Popper function* defined on a finite language is represented by a *truth-table conditional probability function* generated by a *probabilistic ranked truth-table*.

Appendix: For formal statements of theorems and their proofs go to http://faculty-staff.ou.edu/H/James.A.Hawthorne-1/Primer-Appendix.pdf

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