

JAMES HAWTHORNE  
DAVID MAKINSON

# The Quantitative/Qualitative Watershed for Rules of Uncertain Inference

**Abstract.** We chart the ways in which closure properties of consequence relations for uncertain inference take on different forms according to whether the relations are generated in a quantitative or a qualitative manner. Among the main themes are: the identification of watershed conditions between probabilistically and qualitatively sound rules; failsafe and classicality transforms of qualitatively sound rules; non-Horn conditions satisfied by probabilistic consequence; representation and completeness problems; and threshold-sensitive conditions such as ‘preface’ and ‘lottery’ rules.

*Keywords:* nonmonotonic logic, uncertain inference, consequence relations, Horn rules, conditional probabilities, probabilistic thresholds.

## 1. Introduction

Broadly speaking, there are two ways of approaching the formal analysis of uncertain reasoning: quantitatively, using in particular probability relationships, or by means of qualitative criteria. As is widely recognized, the consequence relations that are generated in these two ways behave quite differently.

To be sure, because of the much greater mathematical richness of the real interval  $[0,1]$  compared to the two-element set  $\{0,1\}$ , it is possible to simulate the qualitative approach within variant forms of the quantitative one. This can be done, for example, by admitting infinitesimal values for probability functions, using sequences of standard probability functions approaching unity as a limit, or generalizing to so-called ‘possibility functions’ (see, e.g., Lehmann and Magidor [13], Dubois and Prade [6]). It can also be done using always-defined (e.g. Popper) functions for conditional probability (e.g. Hawthorne [8]), or by working with sums of improbabilities of premises (e.g. Adams [2]).

Moreover, even within the qualitative sphere there is quite a lot of variation in the properties of consequence relations, according to the particular mode of generation. As is well known, default-assumption constructions tend to be the best behaved. Default-valuation constructions using preferential

---

Presented by **Branden Fitelson: Formal Epistemology I**; *Received* June 15, 2006

models are almost as well behaved, and indeed equally so in the finite case. Finally, default-rule constructions are notorious for their irregular behavior, including loss of both disjunction in the premises and cautious monotony. For an overview of such variations see, e.g., Makinson [16], with more details in Makinson [15].

Nevertheless, if we focus on the Horn conditions that are typical of qualitative ways of generating consequence relations, we can identify them with those validated by stoppered (alias smooth) preferential models, i.e., those derivable from the well-known family  $\mathbf{P}$  of closure conditions isolated by Kraus, Lehmann and Magidor [10] and recalled below. This family is generally regarded as forming an ‘industry standard’ for qualitative non-monotonic inference.

In contrast, the consequence relations defined using standard probability functions, conditionalization, and thresholds, are much less regular in their behavior, validating relatively few closure conditions. Indeed, only three of the usual six rules constituting the family  $\mathbf{P}$  are sound in unqualified form for probabilistic inference.

The purpose of this paper is to look closely at the gap between the inference rules that are sound under the qualitative and quantitative approaches. The organization is as follows.

- Part I investigates threshold-independent rules for probabilistic consequence. In particular:
  - In section 2 we recall the qualitative notion of consequence in terms of stoppered preferential models, and the corresponding family  $\mathbf{P}$  of closure conditions; likewise the notion of probabilistic consequence and the family  $\mathbf{O}$  of closure conditions sound for probabilistic consequence.
  - Section 3 discusses the notions of the failsafe and classicality transforms of qualitatively sound Horn rules.
  - Section 4 shows the special role of the rule AND as a watershed condition between qualitatively sound and probabilistically sound rules. It also shows the equivalence of a wide range of systems in their application to a restricted class of Horn rules.
  - Section 5 is devoted to problems of completeness and representation for probabilistically sound families of rules.
  - Section 6 examines the special role of the rule ROR (which is closely related to conditional excluded middle) as a watershed condition for non-Horn rules for systems that extend  $\mathbf{O}$ .

- Part II turns to threshold-sensitive rules. Specifically:
  - Section 7 examines threshold-sensitive Horn rules, notably the ‘pre-ace’ rules introduced by Hawthorne [8].
  - Section 8 generalizes the threshold-sensitive Horn rules described in section 7.
  - Section 9 investigates threshold-sensitive non-Horn rules, considering such non-Horn conditions as the ‘lottery’ rules introduced by Hawthorne [8], and generalizing them.
  - Finally, section 10 gathers together a number of open problems that were raised in the text.

## Part I. Qualitative Rules versus Threshold-Independent Probabilistic Rules

### 2. Qualitative and Probabilistic Consequence Relations, and the Families P and O

By a consequence relation we mean a relation  $\vdash$  between classical propositional formulae. No specific formal conditions are required of it to merit the name, but the relation is understood as intended to represent some notion of inferability, certain or uncertain, between propositions.

The most widely used qualitative notion of uncertain consequence is that formulated by Kraus, Lehmann and Magidor [10] in terms of stoppered (alias smooth) preferential models.

*Definition 2.1.* A *preferential model* is a structure  $(S, <, \models)$  where  $S$  is an arbitrary set (whose elements are called *states*),  $<$  is a transitive, irreflexive relation over  $S$  (called the *preference relation*), and  $\models$  is a *satisfaction relation* between states on the left and propositional formulae on the right, well-behaved with respect to the classical connectives:  $s \models a \wedge b$  iff  $s \models a$  and  $s \models b$  and so on for the other connectives. (Intuitively,  $\models$  specifies which formulae are true in the state  $s$ .)

The preferential model is called *stoppered* (alias smooth) iff whenever a state  $s$  satisfies a formula  $a$  then either  $s$  is minimal under  $<$  among the states satisfying  $a$  or else there is a state  $s' < s$  that is minimal under  $<$  among the states satisfying  $a$ .

Every stoppered preferential model  $(S, <, \models)$  determines a *preferential consequence relation*  $\sim$  by the rule:  $a \sim x$  iff  $s$  satisfies  $x$  for every state  $s \in S$  that is minimal among those satisfying  $a$ .

For further background, see either Kraus, Lehmann and Magidor [10], or a textbook presentation such as that in chapter 3 of Makinson [16].

Probabilistic consequence relations are defined in an entirely different manner. Let  $p$  be any finitely additive probability function in the standard sense, i.e. any function on the set of all classical propositional formulae into the real interval  $[0,1]$  satisfying the Kolmogorov postulates. Let  $a$  be any propositional formula with  $p(a) \neq 0$ . We write  $p_a$  for the conditionalization of  $p$  on  $a$ , i.e. for the probability function defined by the standard equation  $p_a(x) = p(a \wedge x)/p(a)$ .<sup>1</sup> Probabilistic consequence relations are then defined as follows.

*Definition 2.2.* Let  $p$  be any probability function, and let  $t$  be any real number in the interval  $[0,1]$ . The pair  $p,t$  generates a probabilistic consequence relation  $\sim_{p,t}$  (briefly ' $\sim$ ' when no ambiguity is possible) by the rule:  $a \sim x$  iff either  $p(a) = 0$  or  $p_a(x) \geq t$ . The parameter  $t$  is called the *threshold* associated with  $p$  for the relation  $\sim$ .

The behavior of the qualitatively defined consequence operations is very well understood since the seminal work of Kraus, Lehmann and Magidor, briefly KLM, [10], who showed that the class of consequence relations generated by stoppered preferential models may be characterized by a set of syntactic conditions. This result is known as the KLM representation theorem. The syntactic conditions used are all *closure conditions*, alias *Horn rules* with finitely many premises, i.e. conditions of the form: whenever  $a_1 \sim x_1$  and  $\dots$  and  $a_n \sim x_n$  then  $b \sim y$ , where  $n \geq 0$ , possibly with side-conditions  $b_1 \vdash y_1, \dots, b_m \vdash y_m$  where  $\vdash$  is classical consequence.

Whenever we speak simply of Horn rules in sections 1 through 4, we will always mean ones with finitely many premises. Infinite-premise Horn rules will not appear until section 5 (Corollary 5.2), and from then onwards we will distinguish them explicitly.

---

<sup>1</sup>The reasons for using the notation  $p_a$  rather than the more common one  $p(x|a)$  are explained in Makinson [16] chapter 5. However nothing in the paper depends on this notational preference. We are using standard (unconditional) probability functions, and defining conditional probability by the usual ratio condition  $p_a(x) = p(a \wedge x)/p(a)$ , so that  $p_a$  is undefined when  $p(a) = 0$ . We note, however, that most of our results continue to hold when probability is defined as a two-place function defined on all argument pairs satisfying the Popper postulates (see, e.g., Hawthorne [8]).

*Definition 2.3.* The family  $\mathbf{P}$  is made up of the following closure conditions on a consequence relation:

$a \vdash a$  (REFLEX: reflexivity)

Whenever  $a \vdash x$  and  $x \vdash y$ , then  $a \vdash y$  (RW: right weakening)

Whenever  $a \vdash x$  and  $a \dashv\vdash b$ , then  $b \vdash x$  (LCE: left classical equivalence)

Whenever  $a \vdash x \wedge y$ , then  $a \wedge x \vdash y$  (VCM: very cautious monotony)

Whenever  $a \vdash x$  and  $b \vdash x$ , then  $a \vee b \vdash x$  (OR: disjunction in the premises)

Whenever  $a \vdash x$  and  $a \vdash y$ , then  $a \vdash x \wedge y$  (AND: conjunction in conclusion).

Here  $\dashv\vdash$  is classical equivalence. Clearly, given the rule *right weakening*, *reflexivity* is equivalent to the rule called *supraclassicality*, to which we will often appeal in derivations.

Whenever  $a \vdash x$ , then  $a \vdash x$  (SUP: supraclassicality)

Reflexivity is clearly a zero-premise rule. We also treat supraclassicality as a zero-premise rule, in the sense that it has no premise containing  $\vdash$ ; the proviso  $a \vdash x$  is regarded as a side-condition. By the same token, RW and LCE are regarded as one-premise rules with side-conditions. In the literature LCE, left classical equivalence, is often called LLE, left logical equivalence.

To be precise, KLM [10] formulated the family  $\mathbf{P}$  with one difference: instead of VCM (very cautious monotony) they used the rule:

Whenever  $a \vdash x$  and  $a \vdash y$ , then  $a \wedge x \vdash y$  (CM: cautious monotony).

Clearly, in the presence of RW, CM implies VCM; and conversely, with AND available, VCM implies CM. So the two are interchangeable in family  $\mathbf{P}$ . But for weaker systems without AND this is not so. The presentation of  $\mathbf{P}$  using VCM will prove helpful as we probe some characteristic differences between  $\mathbf{P}$  and a weaker family of rules for probabilistic inference.

The well-known condition of cumulative transitivity (CT), alias CUT deserves separate consideration. It is a converse of cautious monotony:

Whenever  $a \vdash x$  and  $a \wedge x \vdash y$ , then  $a \vdash y$  (CT: cumulative transitivity).

As noted by KLM [10], this is derivable in  $\mathbf{P}$ . For suppose  $a \sim x$  and  $a \wedge x \sim y$ . Then,  $a \wedge x \sim \neg x \vee y$  (RW). Also,  $a \wedge \neg x \sim \neg x \vee y$  (SUP). Hence  $(a \wedge x) \vee (a \wedge \neg x) \sim \neg x \vee y$  (OR), so  $a \sim \neg x \vee y$  (LCE). Putting this together with  $a \sim x$  and applying AND gives  $a \sim (\neg x \vee y) \wedge x$  and so by RW again,  $a \sim y$ .

Which of these closure conditions are satisfied when consequence is understood probabilistically as in Definition 2.2 above? We recall the following well-known facts.

*Observation 2.1.* Probabilistic consequence relations satisfy the conditions REFLEX, LCE, RW.

*Observation 2.2.* Probabilistic consequence relations do not always satisfy the conditions AND, OR. Nor do they always satisfy the derived rules CM, CT. Indeed, more strongly: for each one of these conditions and for every threshold  $t$  such that  $0 < t < 1$ , there is a probability function  $p$  that violates that condition.

Another family of closure conditions, specifically tailored for probabilistic consequence, was introduced by Hawthorne [8], c.f. also Hawthorne [9].

*Definition 2.4.* The family  $\mathbf{O}$  of closure conditions on a consequence relation is made up of REFLEX, RW, LCE, VCM and the following weakened versions of OR, AND:<sup>2</sup>

Whenever  $a \wedge b \sim x$  and  $a \wedge \neg b \sim x$ , then  $a \sim x$  (WOR, i.e. weak OR)

Whenever  $a \sim x$  and  $a \wedge \neg y \sim y$ , then  $a \sim x \wedge y$  (WAND, i.e. weak AND).

WOR is immediately derivable in family  $\mathbf{P}$  using LCE, since  $a$  is classically equivalent to  $(a \wedge b) \vee (a \wedge \neg b)$ . In the context of the other rules of  $\mathbf{O}$ , WOR is equivalent to the following rule, which is more directly comparable to OR:

---

<sup>2</sup>As defined in Hawthorne [8] and [9],  $\mathbf{O}$  contains one more rule which may either be stated as “for some  $a$  and  $x$ ,  $a \not\sim x$ ” or as “ $\top \not\sim \perp$ ”. This rule eliminates only one trivial consequence relation – the one where for all  $a$  and  $x$ ,  $a \sim x$ . It is a non-Horn rule, and makes no difference to our treatment here, so we omit it in this paper. Hawthorne’s [8] rules for  $\mathbf{O}$  also differ from those given here in another minor respect. There the pair of rules ‘if  $a \sim x$  then  $a \sim a \wedge x$ ’ (called “Weak And” there, and referred to below as “Antecedence”) and ‘if  $a \sim x$  and  $\neg y \sim y$  then  $a \sim y \wedge x$ ’ (called “Conjunctive Certainty” there) are employed in place of the rule WAND presented here and in Hawthorne [9]. Given REFLEX, RW, LCE, and VCM (which are common to both presentations), this pair of rules is interderivable with WAND.

Whenever  $a \sim x$ ,  $b \sim x$  and  $\vdash \neg(a \wedge b)$ , then  $(a \vee b) \sim x$  (XOR: exclusive OR).

The equivalence is easy to verify. First suppose WOR, and suppose the antecedent of XOR. Then from  $a \sim x$  by LCE,  $(a \vee b) \wedge a \sim x$ . Also the supposition  $\vdash \neg(a \wedge b)$  implies that  $b \dashv\vdash (a \vee b) \wedge \neg a$ , so from  $b \sim x$  we get  $(a \vee b) \wedge \neg a \sim x$  by LCE. Thus,  $(a \vee b) \sim x$  by WOR. For the converse suppose XOR, and suppose the antecedent of WOR. Then  $(a \wedge b) \vee (a \wedge \neg b) \sim x$  by XOR, so  $a \sim x$  by LCE.

WAND is also derivable in family **P**: supposing  $a \sim x$  and  $a \wedge \neg y \sim y$ , by SUP we have  $a \wedge y \sim y$ , so using WOR we get  $a \sim y$ , and thus from AND we have  $a \sim x \wedge y$ . We will look more closely at the rather unusual condition  $a \wedge \neg y \sim y$  in the next section (Observation 3.1).

Hawthorne [8] observed that the rules in family **O** are sound for probabilistic consequence relations when the probability functions employed are Popper functions. They are also clearly sound for standard probability functions.

*Observation 2.3.* (Hawthorne [8], [9]). Probabilistic consequence relations satisfy all the conditions in family **O**.

For the convenience of the reader we gather the verification of this and some other known results in the appendix, along with many of the more routine verifications of new results.

Combining this with the preceding observations we see that **O** is indeed properly weaker than **P**.

Observation 2.3 raises important problems of completeness and representation. Is **O** *Horn complete* for probabilistic consequence? In other words, is each Horn rule that is sound for all probabilistic consequence relations derivable from the conditions in **O**? Is there a *representation theorem* for probabilistic consequence in terms of **O**, in the sense that every consequence relation satisfying all rules in **O** is determined by some pair  $p, t$  where  $p$  is a (standard) probability function and  $t$  is a threshold?

These problems are not the same: while a positive answer for representation immediately implies a positive answer for Horn completeness (and indeed for completeness with respect to broader classes of syntactic conditions), the converse need not hold. We will examine these rather subtle problems in section 5. For the present we give the following result on the relation between probabilistic and qualitative soundness. Despite its basic nature, we have not been able to find a proof, or even a statement of it in

the literature.

*Observation 2.4.* Every Horn rule that is probabilistically sound (i.e. holds for all probabilistic consequence relations) is qualitatively sound (i.e. holds for all consequence relations generated by a stoppered preferential model).

*Remark.* Recall that in sections 1 through 4 we are using the term ‘Horn rule’ to mean ‘finite-premise Horn rule’. In Corollary 5.2 we will see that Observation 2.4 fails for countable-premise Horn rules.

PROOF. Consider any instance of a Horn rule, with premises  $a_1 \sim x_1, \dots, a_n \sim x_n$  and conclusion  $b \sim y$ , possibly with side-conditions. Suppose that this instance fails in some stoppered preferential model. We want to show that it is not probabilistically sound.

To simplify the argument that follows, we may assume without loss of generality that the only elementary letters in the language are those occurring in  $a_1, x_1, \dots, a_n, x_n, b, y$  and in the side-conditions, plus fresh letters added in a way to ensure that the preferential model may be chosen to be injective (i.e. distinct states are labeled by distinct Boolean valuations) without disturbing the failure of the rule instance. Details for this are given in chapter 3 of Makinson [16]. Since the preferential model is injective, we may simply identify states with Boolean valuations on the elementary letters.

Since the rule instance fails, there is a minimal  $b$ -valuation that is not a  $y$ -valuation, while for every  $i$  with  $1 \leq i \leq n$ , every minimal  $a_i$ -valuation is an  $x_i$ -valuation. We choose one minimal  $b$ -valuation that is not a  $y$ -valuation and call it  $v_0$ . Without loss of generality, we may assume that the premises  $a_i \sim x_i$  are listed in a convenient order: for some  $m$  with  $0 \leq m \leq n$  we have (1) for all  $i$  with  $1 \leq i \leq m$  there is a minimal  $a_i$ -valuation less than  $v_0$ , (2) for all  $i$  with  $m < i \leq n$  there is no minimal  $a_i$ -valuation less than  $v_0$ .

For each  $i$  with  $1 \leq i \leq m$ , we choose one such minimal  $a_i$ -valuation, calling it  $v_i$ . With each valuation  $v_0, v_1, \dots, v_m$  we associate the unique state-description  $s_0, s_1, \dots, s_m$  on the elementary letters that it satisfies. We define a function  $p$  on all state descriptions in those letters by putting  $p(s_i) = 1/m+1$  for  $0 \leq i \leq m$  while  $p(s) = 0$  for all other state descriptions. Then  $p$  can be extended uniquely to a probability function on all formulae generated by the elementary letters, which for simplicity we also call  $p$ . Choose as threshold  $t = 1/m+1$ . To complete the proof we need only check the following:

$$(1) p(b \wedge y)/p(b) < t.$$



(2) For each  $i$  with  $1 \leq i \leq n$ , either  $p(a_i) = 0$  or  $p(a_i \wedge x_i)/p(a_i) \geq t$ .

For (1): Since  $v_0(b) = 1$  we have  $s_0 \vdash b$  so  $p(b) \geq p(s_0) = 1/m+1 > 0$ . On the other hand, none of  $v_1, \dots, v_m$  satisfies  $b$ , since each such  $v_i < v_0$  and  $v_0$  is by construction a minimal valuation satisfying  $b$ . Since also  $v_0(y) = 0$ , we have  $v_i(b \wedge y) = 0$  for all  $i$  with  $0 \leq i \leq m$ , so  $p(b \wedge y) = 0$  and thus  $p(b \wedge y)/p(b) = 0 < t$  as desired.

For (2): Choose any  $i$  with  $1 \leq i \leq n$ . We break the argument into two cases.

*Case 1.* Suppose  $i > m$ . Note that in this case we cannot have both  $v_0(a_i) = 1$  and  $v_0(x_i) = 0$ . For since  $a_i \vdash x_i$  holds in the preferential model, that would imply that  $v_0$  is not a minimal  $a_i$ -state, so by stoppering there would be an  $a_i$ -state less than  $v_0$ , so that  $i \leq m$  contradicting the conditions of the case. Thus the following two subcases are exhaustive.

*Subcase 1.1.*  $v_0(a_i) = 1 = v_0(x_i)$ . Then  $v_0(a_i \wedge x_i) = 1$  and so  $p(a_i) \neq 0$  and  $p(a_i \wedge x_i) \geq 1/m+1$ , so  $p(a_i \wedge x_i)/p(a_i) \geq 1/m+1 = t$  as desired.

*Subcase 1.2.*  $v_0(a_i) = 0$ . Then  $s_0 \not\vdash a_i$ . It also follows from the conditions of Case 1 that there is no minimal  $a_i$ -valuation that is less than  $v_0$ , so  $a_i$  is not true under any of  $v_1, \dots, v_m$ . Thus none of the state-descriptions  $s_0, s_1, \dots, s_m$  classically implies  $a_i$ , so  $p(a_i) = 0$  as desired.

*Case 2.* Suppose  $i \leq m$ . Then by construction  $v_i$  is a minimal  $a_i$ -valuation, so  $v_i(a_i) = 1$  and so  $p(a_i) \neq 0$ ; also, since the premise  $a_i \vdash x_i$  holds in the preferential model we have  $v_i(x_i) = 1$ , so  $v_i(a_i \wedge x_i) = 1$  so  $p(a_i \wedge x_i) \geq 1/m+1$ , so  $p(a_i \wedge x_i)/p(a_i) \geq 1/m+1 = t$  as needed. ■

*Corollary 2.5.* Every probabilistically sound Horn rule is derivable from family **P**.

PROOF. Stated more fully, the corollary says: every probabilistically sound (finite-premise) Horn rule is satisfied by every consequence relation that satisfies all rules in family **P**. This is immediate from Observation 2.4 and the KLM representation theorem for **P**. ■

### 3. Failsafe and Classicality Transforms of Closure Conditions

The passage from AND to WAND in the definition of the family **O** deserves further comment. One of the two premises of AND,  $a \vdash y$ , is replaced by what we will call its *failsafe version*,  $a \wedge \neg y \vdash y$ . As we have already remarked, given the other rules in **O**, this transformation strengthens the premise of the rule:  $a \vdash y$  follows immediately from  $a \wedge \neg y \vdash y$  by SUP, WOR, and LCE. The transformation thus weakens the rule itself.

The present section examines closely failsafe and related conditions; some readers may prefer to follow the main line of argument in sections 4 and 5, and return to this section as background for section 6.

Intuitively, while  $a \sim y$  expresses the notion that  $a$  provides *good reason* for  $y$ , the stronger  $a \wedge \neg y \sim y$  may be understood as saying that  $a$  provides *certain reason* for  $y$ . We can be more specific if we think in terms of the two kinds of model.

- Qualitatively, in terms of preferential models,  $a \wedge \neg y \sim y$  says that the model contains no minimal  $(a \wedge \neg y)$ -states; so for stoppered preferential models it says that the model has no  $(a \wedge \neg y)$ -states at all, i.e. all of the states of the model satisfying  $a$  also satisfy  $y$  (see Makinson [16] for background).
- Quantitatively, for  $\sim$  defined with probability function  $p$  and threshold  $t$  as parameters as in Definition 2.2 above:  $a \wedge \neg y \sim y$  holds iff either  $t = 0$  or  $p(a \wedge \neg y) = 0$ , i.e. iff either  $t = 0$  or  $p(a \wedge y) = p(a)$ , i.e. iff either  $t = 0$  or  $p(a) = 0$  or  $p_a(y) = 1$ .

The failsafe condition  $a \wedge \neg y \sim y$  may be given several formulations that are equivalent modulo the family  $\mathbf{O}$  of rules (c.f. Bochman [4] chapter 6).

*Observation 3.1.* In  $\mathbf{O}$  the following are all equivalent: (1)  $a \wedge \neg y \sim y$ , (2)  $a \wedge \neg y \sim \perp$ , (3)  $a \wedge \neg y \sim z$  for all  $z$ , (4)  $(a \wedge b) \wedge \neg y \sim y$  for all  $b$ , (5)  $a \wedge b \sim y$  for all  $b$ .

PROOF. We cycle around the five. Suppose (1): then by LCE,  $(a \wedge \neg y) \wedge \neg y \sim y$ , and clearly also by SUP,  $a \wedge \neg y \sim \neg y$ , so by WAND  $a \wedge \neg y \sim \neg y \wedge y$ , giving by RW  $a \wedge \neg y \sim \perp$ , i.e. (2). Suppose (2): then by RW,  $a \wedge \neg y \sim z$  for all  $z$ , i.e. (3). Suppose (3): then in particular  $a \wedge \neg y \sim b \wedge y$ , so by VCM  $(a \wedge \neg y) \wedge b \sim y$  and thus by LCE  $(a \wedge b) \wedge \neg y \sim y$ . Suppose (4): since by SUP we have  $(a \wedge b) \wedge y \sim y$ , we may apply WOR and LCE to get  $a \wedge b \sim y$ . Suppose (5): then instantiating  $\neg y$  for  $b$ , we have  $a \wedge \neg y \sim y$  and the cycle is complete. ■

It is convenient to introduce a special sign for the relation  $\approx$  of *certain reason*, defined by putting  $a \approx y$  iff  $a \wedge \neg y \sim y$ . From Observation 3.1 we see that in the context of  $\mathbf{O}$ ,  $\approx$  has two distinct monotonicity properties. On the one hand (using the passage from (1) to (5) in Observation 3.1) whenever *certain reason* holds, *good reason* holds monotonically:  $a \approx y$  implies  $a \wedge b \sim y$  for all  $b$ . On the other hand (using the passage from (1)

to (4) in the Observation) the relation of *certain reason* is itself monotonic:  $a \approx y$  implies  $a \wedge b \approx y$  for all  $b$ .

As well as being monotone, the relations  $\approx$  satisfy all rules in  $\mathbf{P}$  so long as  $\sim$  satisfies all rules in  $\mathbf{O}$ , as may be verified easily. Thus, if we define the canonical compact extension of  $\approx$  to arbitrary sets of formulae on the left in the natural way (putting  $A \approx x$  iff  $a \approx x$  for some conjunction  $a$  of finitely many elements of  $A$ ) it becomes a compact supraclassical closure relation satisfying disjunction in the premises, and so by Theorem 2.2 of Makinson [16] may be represented as a pivotal consequence relation. That is:

*Observation 3.2.* Let  $\sim$  be any consequence relation satisfying all rules in  $\mathbf{O}$ , and let  $\approx$  be its failsafe counterpart. Then there is a set  $K$  of formulae such for all  $A, x$ , we have  $A \approx x$  iff  $A \cup K \vdash x$ , where  $\vdash$  is classical consequence.

The notion of the failsafe version of a condition reveals an interesting connection between the family  $\mathbf{P}$  and the family  $\mathbf{O}$ . Take any Horn rule. We define its *failsafe transforms* as follows. If the rule has zero or one premise, it is its own failsafe transform. If it has two or more premises,  $a_1 \sim x_1, \dots, a_n \sim x_n$  we define its failsafe transforms to be the  $n$  rules obtained by replacing one of the premises  $a_i \sim x_i$  by its failsafe version  $a_i \approx x_i$  (i.e.  $a_i \wedge \neg x_i \sim x_i$ ). As we will be applying this notion to the family  $\mathbf{P}$  in this section, we need here only consider cases where  $n \leq 2$ ; in sections 6 and 7 we will need to consider cases where  $n > 2$ .

Write  $\text{FS}(\mathbf{P})$  for the family of failsafe transforms of rules in  $\mathbf{P}$ . Thus  $\text{FS}(\mathbf{P})$  consists of the rules: REFLEX, RW, LCE, VCM, FS(OR) (i.e. whenever  $a \sim x$  and  $b \wedge \neg x \sim x$ , then  $a \vee b \sim x$ ), and FS(AND) (i.e. whenever  $a \sim x$  and  $a \wedge \neg y \sim y$ , then  $a \sim x \wedge y$ ). Strictly speaking, each of AND, OR has two failsafe transforms; but as the source rules are symmetric the two transforms are equivalent (given LCE and RW), so we need only bother with one for each.

*Observation 3.3* Every rule in  $\text{FS}(\mathbf{P})$  is derivable from family  $\mathbf{O}$ . But not conversely: WOR is not derivable from  $\text{FS}(\mathbf{P})$ . Indeed, WOR is not derivable by adding failsafe transformations of any Horn rules with two or more premises to REFLEX, RW, LCE, VCM.

PROOF. For the positive part, since FS(AND) is just WAND, we need only show that FS(OR) is derivable from family  $\mathbf{O}$ . Suppose  $a \sim x$  and  $b \wedge \neg x \sim x$ . We need to derive  $a \vee b \sim x$  using only rules in family  $\mathbf{O}$ . By Observation 3.1 (1) to (5) the second supposition gives us  $b \wedge \neg a \sim x$ . Hence by XOR (see discussion just after definition 2.4) and LCE we have  $a \vee b \sim x$  as desired.

For the negative part, we can define a property that holds for REFLEX, RW, LCE, VCM and all failsafe transformations of Horn rules with two or more premises, and is preserved under chaining, but which fails for the rule WOR. An interesting feature of this property is that it is essentially quantitative, although the rules that it separates are purely qualitative. Details are given in the appendix. ■

While a failsafe relation  $a \approx y$  (i.e.  $a \wedge \neg y \sim y$ ) is stronger than its plain counterpart  $a \sim y$ , it is still weaker than the classical consequence relation  $a \vdash y$ . Given  $a \vdash y$ , we have  $a \wedge \neg y \sim y$  immediately by the monotony of classical consequence and SUP. Thus we get an even weaker version of a Horn rule if we replace  $a \sim y$  by  $a \vdash y$  in one of the premises (thus also changing its status from premise to side-condition). In the case of AND this replacement produces the rule: whenever  $a \sim x$  and  $a \vdash y$  then  $a \sim x \wedge y$ .

This idea is mooted by Kyburg, Teng, and Wheeler [12], but we can go further. We get another connection, this time between the family **P** and a family weaker than **O** that has appeared in the literature on several occasions under various names.

Burgess [5] introduced a system that he called ‘basic subjunctive conditional logic’, and proved a completeness theorem for it in terms of Lewis’ structures for counterfactual conditionals. It was further discussed (in an equivalent form) by van Benthem [18], Adams [1], and Bochman [4] chapter 6. We use Bochman’s formulation and his name for it, **B**. It consists of the rules REFLEX, RW, LCE, VCM (shared with **O** and **P**), together with the further rules of ‘deduction’ (whenever  $a \wedge b \sim x$  then  $a \sim b \rightarrow x$ ) and ‘antecedence’ (whenever  $a \sim x$  then  $a \sim a \wedge x$ ).

Take any Horn rule. We define its *classicality transforms* as follows. If it has two or more premises,  $a_1 \sim x_1, \dots, a_n \sim x_n$  they are the  $n$  rules obtained by replacing one of the premises  $a_i \sim x_i$  by its classical counterpart  $a_i \vdash x_i$ . As before, if the rule has zero or one premise, it is its own classicality transform.

Write  $\text{CL}(\mathbf{P})$  for the family of classicality transforms of rules in **P**, again ignoring the copies arising from symmetry. Thus  $\text{CL}(\mathbf{P})$  consists of the rules: REFLEX, RW, LCE, VCM, CL(OR) (i.e. whenever  $a \sim x$  and  $b \vdash x$  then  $a \vee b \sim x$ ), and CL(AND) (i.e. whenever  $a \sim x$  and  $a \vdash y$  then  $a \sim x \wedge y$ ).

Clearly all rules in  $\text{CL}(\mathbf{P})$ , and likewise all rules in **B**, have at most one premise, and they are all derivable from family **O**.

*Observation 3.4.*  $\text{CL}(\mathbf{P})$  is equivalent to family **B**.

PROOF. We need to show that the rules of deduction and antecedence are derivable from family  $\mathbf{CL}(\mathbf{P})$ , and that  $\mathbf{CL}(\mathbf{OR})$  and  $\mathbf{CL}(\mathbf{AND})$  are derivable from family  $\mathbf{B}$ .

For deduction, suppose  $a \wedge b \vdash x$ . We need to derive  $a \vdash b \rightarrow x$  using only rules in family  $\mathbf{CL}(\mathbf{P})$ . By RW on the supposition,  $a \wedge b \vdash b \rightarrow x$ . Also  $(a \wedge \neg b) \vdash b \rightarrow x$ , so by  $\mathbf{CL}(\mathbf{OR})$  we have  $(a \wedge b) \vee (a \wedge \neg b) \vdash b \rightarrow x$ , and thus by (LCE),  $a \vdash b \rightarrow x$  as desired.

For antecedence, suppose  $a \vdash x$ . We need to derive  $a \vdash a \wedge x$  using only rules in family  $\mathbf{CL}(\mathbf{P})$ . Trivially,  $a \vdash a$  so by  $\mathbf{CL}(\mathbf{AND})$  we have immediately  $a \vdash a \wedge x$  as desired.

Conversely, for  $\mathbf{CL}(\mathbf{OR})$ , suppose  $a \vdash x$  and  $b \vdash x$ . We need to derive  $a \vee b \vdash x$  using only rules in family  $\mathbf{B}$ . By LCE on the first supposition we have  $(a \vee b) \wedge (a \vee \neg b) \vdash x$  so by the rule of deduction  $a \vee b \vdash (a \vee \neg b) \rightarrow x$  and thus (RW)  $a \vee b \vdash b \vee x$ . But since  $b \vdash x$  we have  $b \vee x \vdash x$  and so (RW again)  $a \vee b \vdash x$  as desired.

For  $\mathbf{CL}(\mathbf{AND})$ , suppose  $a \vdash x$  and  $a \vdash y$ . We need to derive  $a \vdash x \wedge y$  using only rules in family  $\mathbf{B}$ . By the first supposition antecedence gives  $a \vdash a \wedge x$  so by the second supposition and applying RW,  $a \vdash x \wedge y$  as desired. ■

We will mention the family  $\mathbf{B}$  again in Observation 4.3, but we are much more interested in family  $\mathbf{O}$ , since it is stronger than  $\mathbf{B}$  and we are interested in determining the *strongest* family of probabilistically sound rules.

#### 4. The Powers of AND, OR, CM, CT Modulo O

Of the closure conditions that are qualitatively sound but not quantitatively so, AND appears intuitively to play a special role. The purpose of this section is to make this intuition precise and confirm it. Gathering positive results from Hawthorne [8], Adams [2], and Bochman [4], and adding negative ones where appropriate, we obtain the following picture.

*Observation 4.1.* Modulo the rules in  $\mathbf{O}$ , the rules OR, CT, CM, AND stand in the relationships indicated by Figure 1.

PROOF. The implications from AND were noted by Hawthorne [8]; the converse implication from  $\{\mathbf{CM}, \mathbf{CT}\}$  to AND by Adams [2]; the same implications along with that from  $\{\mathbf{CM}, \mathbf{OR}\}$  to AND by Bochman [4], where the implication from CT to OR is also shown. For the reader's convenience, we recall these verifications (some in variant form) in the appendix (Observations 4.1.1 through 4.1.5).

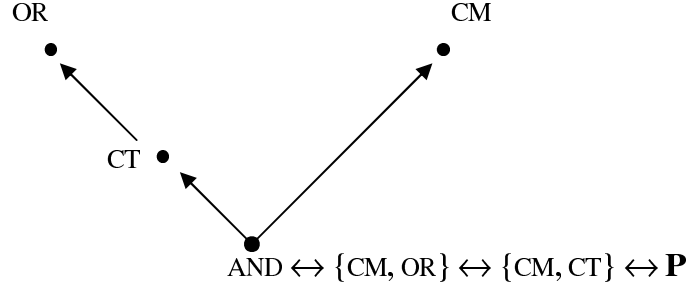


Figure 1. Figure for Observation 4.1

Given the positive implications, it remains to show only the following three non-implications: from CT to AND, from CM to AND, from OR to CT. All others, e.g. from CM to CT, ensue immediately.

The easy non-implication from CT to AND is noted by Bochman [4] (Example 6.7.1), and is shown in the appendix. The remaining two are as follows.

*Observation 4.1.6.* AND is not derivable from  $\mathbf{OU}\{\text{CM}\}$ .

*Verification.* It is surprisingly tricky to find a model illustrating this. The simplest one that we have found satisfying  $\mathbf{OU}\{\text{CM}\}$  but not AND is built over the eight-element Boolean algebra. For simplicity, we identify the eight-element Boolean algebra with the power set of  $\{1,2,3\}$  so that also the Boolean relation  $\leq$ , interpreting classical consequence  $\vdash$ , coincides with the subset relation over  $\{1,2,3\}$ . To reduce tedious notation, we write subsets without braces or commas; for example  $\{1,2\}$  is written as 12, and  $1 \leq a \wedge x$  means  $\{1\} \leq a \wedge x$ , i.e.  $\{1\} \subseteq a \cap x$ .

We define a relation  $\sim$  over this algebra as follows:  $a \sim x$  iff either  $a \leq x$  or else (both  $1 \leq a \wedge x$  and either  $a \neq 123$  or  $x \neq 1$ ). In other words:  $a \sim x$  iff either  $a \leq x$  or else one of the following six conditions holds:  $a = 123$  and  $x = 12$ ,  $a = 123$  and  $x = 13$ ,  $a = 12$  and  $x = 13$ ,  $a = 13$  and  $x = 12$ ,  $a = 12$  and  $x = 1$ ,  $a = 13$  and  $x = 1$ . Note that this list does not include  $a = 123$  and  $x = 1$ ; the relation is not transitive.

The construction may be visualized with Figure 2. Without the arrowheads, we have the Hasse diagram for the Boolean relation  $\leq$ ; the arrowheads indicate the supplements.

In the appendix we verify that this model has the properties claimed.

*Observation 4.1.7.* CT is not derivable from  $\mathbf{OU}\{\text{OR}\}$ .

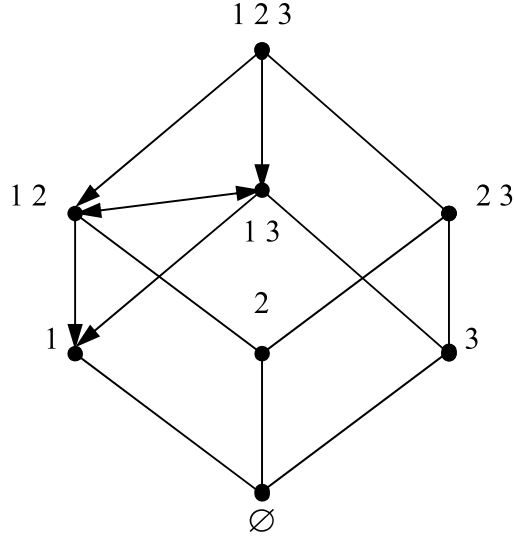


Figure 2. Figure for Observation 4.1.6

*Verification.* The eight-element model used for the preceding Observation fails OR, and so cannot be used for the task. It is possible to construct a variant eight-element model that does work, but the following argument is less tedious.

By a valuation we mean a Boolean valuation defined on all elementary letters of the language. We say that valuations  $v, v'$  are *almost identical*, and write  $v' =_1 v$ , iff they differ on at most one elementary letter. Let  $\vdash$  be the consequence relation defined by putting  $a \vdash x$  iff for every valuation  $v$ , if  $v(a) = 1$  then there is a valuation  $v' =_1 v$  with  $v'(a \wedge x) = 1$ . We claim that this relation satisfies all rules in  $\mathbf{OU}\{\text{OR}\}$  but fails CT.

For the failure of CT, take three distinct elementary letters  $p, q, r$  and note that  $p \vdash q, p \wedge q \vdash p \wedge q \wedge r$ , but  $p \not\vdash p \wedge q \wedge r$ .

On the other hand, REFLEX, RW, LCE trivially succeed. It remains to verify OR, VCM, WAND.

For OR (and thus also WOR), suppose  $a \vdash x$  and  $b \vdash x$ , and suppose also  $v(a \vee b) = 1$ . Then either  $v(a) = 1$  or  $v(b) = 1$ . In each case, there is a  $v' =_1 v$  with  $v'((a \vee b) \wedge x) = 1$ .

For VCM, suppose  $a \vdash x \wedge y$ , and suppose also  $v(a \wedge x) = 1$ . Then  $v(a) = 1$  so there is a  $v' =_1 v$  with  $1 = v'(a \wedge (x \wedge y)) = v'((a \wedge x) \wedge y)$  as needed.

For WAND, suppose  $a \vdash x$  and  $a \wedge \neg y \vdash y$ . First observe that  $a \vdash y$ : for if  $w(a) = 1$  while  $w(y) = 0$  then  $w(a \wedge \neg y) = 1$  so by the second supposition

there is a  $w' =_1 w$  with  $w'((a \wedge \neg y) \wedge y) = 1$ , which is impossible. Now suppose  $v(a) = 1$ . Then by the first supposition there is a  $v' =_1 v$  with  $v'(a \wedge x) = 1$ , so since  $a \vdash y$ , we have  $v'(a \wedge (x \wedge y)) = 1$  as needed. ■

We end this section with a result on Horn rules of a special kind: those with at most one premise and, more generally, finite-premise Horn rules whose premises have pairwise inconsistent antecedents. The general point we shall make is that a broad range of quantitative and qualitative contexts, both semantic and syntactic, identify the same privileged family of these rules.

*Observation 4.3.* Consider any finite-premise Horn rule, from premises  $a_i \sim x_i$  (for  $i \leq n$ ) to conclusion  $b \sim y$  such that the antecedents  $a_i$  of the premises are pairwise inconsistent. Then the following seven conditions are equivalent:

- (1) It is probabilistically sound.
- (2a) It is qualitatively sound (i.e. sound in all stoppered preferential models).
- (2b) It is sound in all linear preferential models containing at most two states.
- (3) It satisfies Adams' truth-table test.
- (4a) It is derivable from family  $\mathbf{B} \cup \{\text{WOR}\}$ , indeed when  $n \leq 1$ , from  $\mathbf{B}$  alone.
- (4b) It is derivable from family  $\mathbf{O}$ .
- (4c) It is derivable from family  $\mathbf{P}$ .

We first explain the meaning of clause (3), relate the various clauses to results in the literature, establish a lemma, and finally give the proof. Probabilistic soundness, soundness in preferential models, the families  $\mathbf{P}$  and  $\mathbf{O}$ , and the rule WOR are all defined in section 2 above; the family  $\mathbf{B}$  defined in section 3. Adams' truth-table test, formulated in his paper [1], is the following: There is some subset  $I \subseteq \{1, \dots, n\}$  such that both  $b \wedge \neg y \vdash \bigvee_{i \in I} (a_i \wedge \neg x_i)$  and  $\bigvee_{i \in I} (a_i \wedge x_i) \vdash b \wedge y$ . Here we use the convention that the disjunction (resp. conjunction) of the empty set of formulae is  $\perp$  (resp.  $\neg \perp$ ). With this reading, when  $n = 0$  the test reduces to  $b \vdash y$ . For  $n = 1$ , it reduces to: Either  $b \vdash y$  or both  $a \rightarrow x \vdash b \rightarrow y$  and  $a \wedge x \vdash b \wedge y$ , where  $a \sim x$  is the sole premise and  $b \sim y$  is the conclusion of the rule. Using just



classical logic, this may in turn be simplified a little: Either  $b \vdash y$  or both  $\neg a \vdash b \rightarrow y$  and  $a \wedge x \vdash b \wedge y$ .

A paraphrase provided by Adams helps give the test an intuitive meaning: For some subset of the premises of the Horn rule we have both (1) falsification of the conclusion classically implies falsification of at least one premise in the subset, and (2) fulfilment (alias verification) of at least one premise in the subset classically implies fulfilment of the conclusion. Adams [1] also claims the equivalence of (1), (3) and (4a) (the last in terms of a system trivially equivalent to **B**). However, in the judgement of the present authors, the proof of equivalence sketched in section D of the appendix of the paper contains a serious gap.

Earlier, van Benthem ([18], Theorem 10.3) showed that in the even more restrictive case of *at most one-premise* Horn rules, derivability from (a certain family equivalent to) **P** implies derivability from the weaker family **B**. His argument makes use of disjunctive normal forms and Lewis models for counterfactual conditionals. Bochman ([4], Corollary 7.5.6) obtained the same result via more general results for **B**.

The following lemma about linear preferential models with at most two elements will facilitate the proof of the implication (2b)  $\Rightarrow$  (3).

*Lemma 4.2.* Consider any finite-premise Horn rule, from premises  $a_i \sim x_i$  (for  $i < m \geq 1$ ) to conclusion  $b \sim y$ , such that the antecedents  $a_i$  of the premises are pairwise inconsistent. Suppose that the rule fails in some linear preferential model with at most two elements. If  $b \wedge \neg y \vdash \bigvee_{i < m} (a_i)$ , then every Horn rule formed by adding another premise  $a_m \sim x_m$  with  $a_m$  inconsistent with each  $a_i$  ( $i < m$ ) also fails in some linear preferential model with at most two elements.

PROOF. (Proof of Lemma 4.2.) By the supposition, there is a linear preferential model of at most two states in which the  $a_i \sim x_i$  ( $i < m$ ) hold but  $b \sim y$  fails, so in this model there is a valuation  $v$  that is a minimal  $b$ -state but is not a  $y$ -state, while every minimal  $a_i$ -state ( $i < m$ ) of the model is an  $x_i$ -state. Using the supposition  $b \wedge \neg y \vdash \bigvee_{i < m} (a_i)$  it follows that  $v(a_k) = 1$  for some  $k < m$ , so by the pairwise inconsistency assumption,  $v(a_i) = 0$  for all  $i < m$ ,  $i \neq k$ . Now consider the addition of a premise  $a_m \sim x_m$  with  $a_m$  inconsistent with each  $a_i$  ( $i < m$ ). Then also  $v(a_m) = 0$ . We split into three cases.

*Case 1.* Suppose that  $v$  is the only state in the preferential model. Then  $a_m \sim x_m$  holds in the model since  $v(a_m) = 0$ , and thus the enlarged Horn rule fails in the model, as desired.

*Case 2.* Suppose that there is a second state  $w$  that does not satisfy  $a_m$ . Then  $a_m \not\sim x_m$  holds in the model since  $a_m$  fails in both states, and again the enlarged Horn rule fails in the model.

*Case 3.* Suppose that the second state  $w$  does satisfy  $a_m$ . Since  $a_m$  is inconsistent with each  $a_i$  ( $i < m$ ) we have  $w(a_k) = 0$ , so  $v$  is a minimal  $a_k$ -state and so also  $v(x_k) = 1$ . Form a one-state linear preferential model by dropping  $w$ , leaving  $v$ . Now  $a_m \not\sim x_m$  holds in this model since  $v(a_m) = 0$ . Also the  $a_i \not\sim x_i$  ( $i < m$ ) continue to hold, because the only  $a_i$  that  $v$  satisfies is  $a_k$  and we know that  $v(x_k) = 1$ . Since also  $v(b) = 1$  while  $v(y) = 0$  we conclude that the enlarged Horn rule fails in this one-state preferential model, as desired. ■

PROOF. Proof of Observation 4.3.

The proof begins with the equivalence (2a)  $\Leftrightarrow$  (4c), then shows how to cycle around the first six conditions: (1)  $\Rightarrow$  (2a)  $\Rightarrow$  (2b)  $\Rightarrow$  (3)  $\Rightarrow$  (4a)  $\Rightarrow$  (4b)  $\Rightarrow$  (1).

The equivalence (2a)  $\Leftrightarrow$  (4c) in fact holds for rules of any kind, Horn or otherwise with any number of premises (and without the pairwise inconsistency assumption), as follows immediately from the representation theorem of Kraus, Lehmann and Magidor [10] (see section 2 above and section 6 below). The implication (2a)  $\Rightarrow$  (2b) clearly also holds without any restrictions.

Two further implications in the cycle hold for Horn rules with any number (finite or infinite) of premises: (4a)  $\Rightarrow$  (4b) follows from the fact that that WOR and every rule in **B** are derivable in family **O** as noted in section 3, and (4b)  $\Rightarrow$  (1) follows from Observation 2.3. The implication (1)  $\Rightarrow$  (2a) holds for Horn rules with any finite number of premises, by Observation 2.4. None of these implications need the pairwise inconsistency assumption.

It remains to show (2b)  $\Rightarrow$  (3) and (3)  $\Rightarrow$  (4a), both of which do need the pairwise inconsistency assumption. We begin with the easier one, (3)  $\Rightarrow$  (4a).

Suppose that a finite-premise Horn rule from premises  $a_i \not\sim x_i$  ( $i \leq n$ ) to conclusion  $b \not\sim y$  satisfies (3); suppose moreover that the  $a_i$  are pairwise inconsistent. Now, suppose that  $a_i \not\sim x_i$  holds for all  $i \leq n$ . We need to derive  $b \not\sim y$  using only rules from family **B** if the rule contains no more than one premise (i.e. if  $n \leq 1$ ), and using only rules from **BU**{WOR} if it contains more than one premise (i.e. if  $n > 1$ ).

In the case  $n = 0$  we have by (3) that  $b \vdash y$  so immediately  $b \not\sim y$  by SUP. Suppose  $n \geq 1$ . Since each  $a_i \not\sim x_i$  ( $i \leq n$ ), we can apply the rule of antecedence (in **B**) to get  $a_i \not\sim a_i \wedge x_i$  for each  $i \leq n$ , and thus for all

$i \in I \subseteq \{1, \dots, n\}$ . By (3),  $\bigvee_{i \in I} (a_i \wedge x_i) \vdash b \wedge y$  so that  $a_i \wedge x_i \vdash b \wedge y$  for each  $i \in I$ , so we can use RW to get each  $a_i \vdash b \wedge y$ . In the case that  $n = 1$  we thus already have  $\bigvee_{i \in I} (a_i) \vdash b \wedge y$ ; in the case  $n > 1$  the pairwise inconsistency of the  $a_i$  allows multiple applications of XOR (which follows from WOR - see the remarks following Definition 2.4) to get the same. Hence by VCM  $\bigvee_{i \in I} (a_i) \wedge b \vdash y$ , so by LCE and the rule of deduction (in **B**) we have  $b \vdash \bigvee_{i \in I} (a_i) \rightarrow y$ . On the other hand, also by (3),  $b \wedge \neg y \vdash \bigvee_{i \in I} (a_i \wedge \neg x_i)$  so  $b \wedge \neg y \vdash \bigvee_{i \in I} (a_i)$ ; contraposing and transforming we have  $\neg \bigvee_{i \in I} (a_i) \vdash b \rightarrow y$ . This (together with  $y \vdash b \rightarrow y$ ) yields  $\bigvee_{i \in I} (a_i) \rightarrow y \vdash b \rightarrow y$ . Hence, from  $b \vdash \bigvee_{i \in I} (a_i) \rightarrow y$ , by RW we have  $b \vdash b \rightarrow y$ . Applying antecedence again to this,  $b \vdash b \wedge (b \rightarrow y)$  and thus finally  $b \vdash y$  (by RW) as desired. This completes the verification of (3)  $\Rightarrow$  (4a).

It remains to show (2b)  $\Rightarrow$  (3). Suppose that a finite-premise Horn rule from premises  $a_i \vdash x_i$  ( $i \leq n$ ) to conclusion  $b \vdash y$  satisfies (2b); suppose moreover that the  $a_i$  are pairwise inconsistent. We need to show that the rule satisfies Adams' truth-table test. The crucial first step is to choose a suitable  $I \subseteq \{1, \dots, n\}$ : we take it to be any *minimal* subset of  $\{1, \dots, n\}$  such that the rule from premises  $a_i \vdash x_i$  ( $i \in I$ ) to conclusion  $b \vdash y$  satisfies (2b). Clearly such a minimal set exists, and to simplify notation without loss of generality we may suppose that  $I = \{1, \dots, m\}$  where  $0 \leq m \leq n$ . We call the corresponding rule, with premises  $a_i \vdash x_i$  ( $i \leq m$ ) and conclusion  $b \vdash y$  the *minimal rule*.

First, we verify  $b \wedge \neg y \vdash \bigvee_{i \leq m} (a_i \wedge \neg x_i)$ , which is the first part of the truth-table test. Suppose otherwise. Then there is a Boolean valuation  $v$  with  $v(b) = 1$ ,  $v(y) = 0$ , and  $v(a_i \rightarrow x_i) = 1$  for all  $i \leq m$ . Consider the one-element linear preferential model whose only state is  $v$ . Then  $a_i \vdash x_i$  holds in this model for all  $i \leq m$ , while  $b \vdash y$  fails in it, contrary to the construction of the minimal rule.

Next we show the second half of Adams' truth-table test, i.e. that  $\bigvee_{i \leq m} (a_i \wedge x_i) \vdash b \wedge y$ . Suppose otherwise; we get a contradiction. By the supposition there is a Boolean valuation  $v$  with  $v(b \wedge y) = 0$  and  $v(a_j \wedge x_j) = 1$  for some  $j \leq m$ . Without loss of generality, we simplify notation by taking  $j = m$ , so  $v(a_m \wedge x_m) = 1$ . By the pairwise inconsistency assumption, since  $v(a_m) = 1$  we have  $v(a_i) = 0$  for all  $i < m$ . We split into two cases.

*Case 1.* Suppose  $v(b) = 1$ . Then since  $v(b \wedge y) = 0$  we have  $v(y) = 0$ . Take the linear preferential model whose sole state is  $v$ . Then  $b \vdash y$  fails,  $a_m \vdash x_m$  succeeds since  $v(a_m \wedge x_m) = 1$ , and also  $a_i \vdash x_i$  (all  $i < m$ ) since  $v(a_i) = 0$  for all  $i < m$ . Thus the minimal rule fails condition (2b) contrary to its construction.

*Case 2:* Suppose  $v(b) = 0$ . Since the set  $\{1, \dots, m\}$  is minimal in the

sense specified and the  $a_i$  ( $i \leq m$ ) are pairwise inconsistent, we may apply Lemma 4.2 to conclude  $b \wedge \neg y \not\vdash \bigvee_{i < m} a_i$ . Hence there is a valuation  $w$  with  $w(b) = 1$ ,  $w(y) = 0$  and  $w(a_i) = 0$  for all  $i < m$ . Take the two-element linear preferential model whose bottom state is labeled with  $v$ , and whose top state is labeled with  $w$ . Since  $w(b) = 1$ ,  $w(y) = 0$ , and  $v(b) = 0$ , we know that  $b \sim y$  fails. Also, since  $v(a_m \wedge x_m) = 1$  and  $v$  is bottom state, we have  $a_m \sim x_m$ . By the pairwise inconsistency assumption,  $v(a_i) = 0$  for all  $i < m$ . But also  $w(a_i) = 0$  for all  $i < m$ , so vacuously  $a_i \sim x_i$  for all  $i < m$ . Putting these together,  $a_i \sim x_i$  for all  $i \leq m$ , and the minimal rule fails condition (2b) contrary to its construction. ■

*Corollary 4.4.* The set of all probabilistically sound finite-premise Horn rules with pairwise inconsistent antecedents to their premises is decidable.

PROOF. Clearly, each of the equivalent conditions (2b) and (3) provides a decision procedure. ■

## 5. Problems of Representation and Completeness

Is there a *representation theorem* for probabilistic consequence in terms of  $\mathbf{O}$ , in the sense that every consequence relation satisfying all rules in  $\mathbf{O}$  is determined by some pair  $p, t$  where  $p$  is a (standard) probability function and  $t$  is a threshold? We begin with a very general negative result concerning representation, and then discuss the problem of completeness.

In earlier sections we have been simply writing ‘Horn rule’ to mean “Horn rule with finitely many premises”. We will now explicitly say “finite premise Horn rule” when we intend this, as we will be dealing with both countable-premise and finite-premise Horn rules.

*Observation 5.1.* Probabilistic consequence is not representable in terms of any family of probabilistically sound finite-premise Horn rules. That is, given any family of probabilistically sound finite-premise Horn rules, there is a consequence relation  $\sim$  that satisfies all of these rules but is not a probabilistic consequence relation – i.e.  $\sim$  cannot be generated by any pair  $p, t$  consisting of a probability function  $p$  and a threshold  $t$ . Moreover, such a  $\sim$  may always be chosen as the consequence relation determined by some linear stoppered preferential model.

Note that from the first part of the theorem it follows that neither the family  $\mathbf{O}$ , nor any extension of it by further probabilistically sound finite-premise Horn rules, gives us a representation theorem. From the second part it follows, for example, that the addition of the non-Horn (negative premise) rule of negation rationality (from  $a \sim x$  and  $a \wedge b \not\sim x$  to  $a \wedge \neg b \sim x$ ), which is sound for probabilistic consequence and also for linear stoppered preferential models, still does not get us a representation theorem.

PROOF. Consider any family of rules of the kind described. We construct in Figure 3 a stoppered preferential model that satisfies it, but which is not determined by any probability function  $p$  and threshold  $t$ .

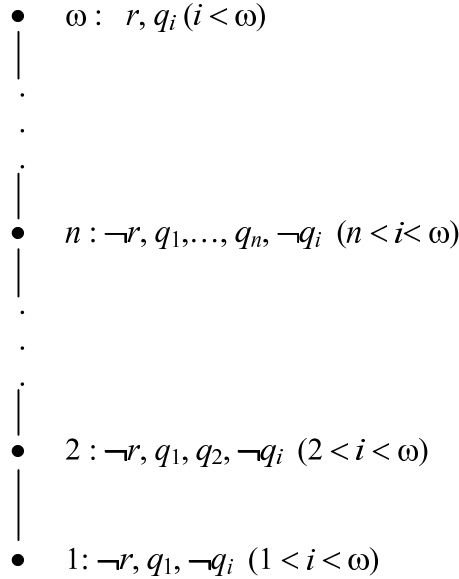


Figure 3. Figure for Observation 5.1

Here the propositional language has elementary letters  $r, q_1, q_2, \dots$  (to avoid confusion we use ‘ $p$ ’ only for probability functions). The states of the preferential model are  $1, 2, \dots, \omega$  with the natural order. Each state is labeled in the diagram by the literals (elementary letters and their negations) that it satisfies. Clearly this model is both linear and stoppered.

Let  $\sim$  be the consequence relation determined by this linear stoppered preferential model. Since all the finite-premise Horn rules in the family are by hypothesis probabilistically sound, we know from Observation 2.4. that  $\sim$  satisfies them. It remains to be shown that  $\sim$  is not a probabilistic

consequence relation for any choice of probability function  $p$  and threshold  $t$ . To prepare for that, we note some features of  $\vdash$ .

Put  $a = r$ ,  $a_i = q_1 \wedge q_2 \wedge \dots \wedge q_i$  ( $1 \leq i < \omega$ ) and  $x_i = a_i \wedge \neg q_{i+1} = q_1 \wedge q_2 \wedge \dots \wedge q_i \wedge \neg q_{i+1}$  ( $1 \leq i < \omega$ ). Then we have the following:

1.  $a \not\vdash \perp$  (since  $\omega$  is a minimal  $a$ -state).
2.  $a \vdash a_i$  for all  $1 \leq i < \omega$  (since  $\omega$  is the unique least  $a$ -state and all  $q_i$  are true at  $\omega$ ).
3.  $a_i \vdash x_i$  for all  $1 \leq i < \omega$  (since the unique least  $a_i$ -state is  $i$  and  $x_i$  is true at  $i$ ).

Moreover,  $a_i \wedge a_j \vdash \neg(x_i \wedge x_j)$  for each distinct  $i$  and  $j$  (since the  $x_i$  are pairwise inconsistent – because whenever  $i < j$ ,  $x_i$  has  $\neg q_{i+1}$  as a conjunct whereas  $x_j$  has  $q_{i+1}$  as a conjunct). Recalling the relation  $\approx$  of certain reasoning, defined from  $\vdash$  in section 3, we thus have:

4.  $a_i \wedge a_j \approx \neg(x_i \wedge x_j)$  for each distinct  $i$  and  $j$ .

Now take any probability function  $p$  and any threshold  $t \in [0,1]$ . Let  $\sim_{p,t}$  be the probabilistic consequence relation determined by  $(p,t)$ . To complete the proof we show that  $\sim_{p,t}$  cannot simultaneously have all of properties (1) through (4) *no matter how the formulae  $a, a_i, x_i$  are chosen*. Suppose that  $\sim_{p,t}$  does satisfy (1) through (4): we derive a contradiction.

By the remarks preceding Observation 3.1, point (4) implies that for each distinct  $i$  and  $j$ ,  $p(a_i \wedge x_i \wedge a_j \wedge x_j) = 0$  or  $t = 0$ . But (1) implies that  $t > 0$ . So, for each distinct  $i$  and  $j$ ,  $p(a_i \wedge x_i \wedge a_j \wedge x_j) = 0$ .

Also by (1),  $p(a) \neq 0$ , so  $p(a) \cdot t^2 > 0$ . Hence there is an integer  $n \geq 1$  with  $1/n < p(a) \cdot t^2$  so  $n \cdot p(a) \cdot t^2 > 1$ . By (2), each  $p(a \wedge a_i)/p(a) \geq t$  so  $p(a \wedge a_i) \geq p(a) \cdot t$  so *a fortiori*  $p(a_i) \geq p(a) \cdot t > 0$ . So by (3) each  $p(a_i \wedge x_i)/p(a_i) \geq t$  so  $p(a_i \wedge x_i) \geq p(a_i) \cdot t \geq p(a) \cdot t^2$ . Hence using (from the previous paragraph) the fact that  $p((a_i \wedge x_i) \wedge (a_j \wedge x_j)) = 0$ ,  $p(\bigvee\{a_i \wedge x_i : 1 \leq i \leq n\}) = \sum_{\{i: 1 \leq i \leq n\}} p(a_i \wedge x_i) \geq n \cdot p(a) \cdot t^2 > 1$  which is impossible. ■

The argument used to prove this theorem also yields an incompleteness result for infinite-premise Horn rules. This contrasts with the positive result for finite-premise Horn rules in Observation 2.4.

*Corollary 5.2.* There is a countable-premise Horn rule that is probabilistically sound but which is not derivable from any family of finite-premise Horn rules, even when the latter is supplemented by a set of rules (Horn or non-Horn, with finitely or infinitely many premises) that are sound in all linear stoppered preferential models.

PROOF. We can build a suitable Horn rule out of conditions (1) through (4). It is the following ‘Archimedean rule’: whenever  $a \sim a_i$  and  $a_i \vdash x_i$  for all  $1 \leq i < \omega$ , and  $a_i \wedge a_j \approx \neg(x_i \wedge x_j)$  for each distinct  $i$  and  $j$ , then  $a \sim \perp$ . ■

It remains an open question whether **O** is complete for finite-premise probabilistically sound Horn rules. It is likewise an open question whether a representation theorem can be established for family **O** plus negation rationality plus the ‘Archimedean rule’.

## 6. From Probabilistic to Stronger Qualitative Systems with non-Horn Rules

A central theme of this paper is to identify rules that mark a watershed between logics for probabilistically defined consequence relations and those defined qualitatively. We saw (Observation 2.4) that every finite-premise Horn rule that is quantitatively sound is qualitatively so in the sense given by the Kraus-Lehmann-Magidor family **P**, and (Observation 4.1) that the rule AND acts as a watershed, sufficing to pass from the former to the latter.

As is well known, family **P** is not the strongest possible system of qualitative consequence. A stronger one is family **R** (for ‘rational consequence’), determined by the class of all ranked stoppered preferential models; and an even stronger one is family **S** (after Robert Stalnaker, [17]), determined by the class of all linear stoppered preferential models. This prompts the question: How do these families look from a probabilistic perspective? This is the subject of the present section. We will give definitions and results, but omit most of the fairly straightforward verifications.

We note first of all that, in contrast with the situation for Horn rules, there is a (finite-premise) non-Horn rule that is probabilistically sound but not qualitatively so (i.e. not true in all stoppered preferential models). This is the rule of *negation rationality*, NR, already mentioned in passing in section 5.

Whenever  $a \sim x$ , then  $a \wedge b \vdash x$  or  $a \wedge \neg b \vdash x$  (NR).

Its failure in some stoppered preferential models is well-known (see, e.g., Freund [7] or the overview in Makinson [15]), while its probabilistic soundness is easily checked as follows.

*Observation 6.1.* NR is probabilistically sound.

PROOF. Suppose  $a \sim x$ , i.e. either  $p(a) = 0$  or  $p_a(x) \geq t$ . If  $p(a \wedge b) = 0$ , then  $a \wedge b \sim x$  and we are done; and similarly if  $p(a \wedge \neg b) = 0$ . So suppose  $p(a \wedge b) > 0$  and  $p(a \wedge \neg b) > 0$ . Then  $p(a) > 0$ , so  $t \leq p_a(x) = p_{a \wedge b}(x) \cdot p_a(b) + p_{a \wedge \neg b}(x) \cdot p_a(\neg b)$ . Hence it is not possible for both  $p_{a \wedge b}(x) < t$  and  $p_{a \wedge \neg b}(x) < t$  to hold. So either  $a \wedge b \sim x$  or  $a \wedge \neg b \sim x$ . ■

On the other hand, it's also well known that NR is derivable in the qualitatively motivated systems **R** and **S** (see, e.g., the overview in Makinson [15]). We briefly recall the definitions and basic properties of systems **R** and **S** here.

- Syntactically, **R** consists of the conditions in **P** together with the rule of *rational monotony* (RM):

whenever  $a \sim x$ , then  $a \sim \neg b$  or  $a \wedge b \sim x$  (RM).

In the context of **P** this rule implies negation rationality (NR).

- Semantically, **R** is determined by the class of ranked stoppered preferential models (see, e.g., Kraus, Lehmann, Magidor [10], and Lehmann and Magidor [13]).
- Syntactically, **S** consists of the conditions in **P** together with the rule *right or* (ROR):

whenever  $a \sim x \vee y$ , then  $a \sim x$  or  $a \sim y$  (ROR).

In the context of **P** this rule is equivalent to the rule of conditional excluded middle (CEM):

$a \sim x$  or  $a \sim \neg x$  (CEM).

It should be noted, however that in the context of systems such as **O** that lack AND, ROR is stronger than CEM.

- Semantically, **S** is determined by the class of all linear stoppered preferential models (see, e.g., Bezzazi, Makinson, Perez [3]), and thus implies rational monotony (RM). **S** is equivalent to a family proposed by Stalnaker [17] for the logic of counterfactual conditionals.

**S** is about as strong a system as one could ever want in a nonmonotonic logic. Indeed, for most purposes weaker families such as **R** and even **P** seem more appropriate.

Note that in contrast with *negation rationality* (NR), neither *rational monotony* (RM) nor *right or* (ROR) is probabilistically sound. Thus, our interest in probabilistically sound families suggests the following development. Just as the strengthened qualitatively sound families **R** and **S** are gotten by



supplementing qualitatively sound  $\mathbf{P}$  with the non-Horn rules  $\mathbf{RM}$  and  $\mathbf{ROR}$ , respectively, it is natural to supplement  $\mathbf{O}$  with the non-Horn rule  $\mathbf{NR}$  to get a strengthened family of probabilistically sound rules.

*Definition 6.1.* The family  $\mathbf{Q}$  of conditions on a consequence relation is made up of  $\mathbf{O}$  together with  $\mathbf{NR}$  (negation rationality): i.e.,  $\mathbf{Q} = \mathbf{OU}\{\mathbf{NR}\}$ .

Indeed, we can characterize more precisely just how  $\mathbf{S}$  and  $\mathbf{R}$  outrun  $\mathbf{Q}$ . In doing so it will prove useful to consider an additional rule,  $\mathbf{PREF}(1)$ , which will reappear below, in part II, on rules for threshold-sensitive probabilistic consequence.

$\mathbf{PREF}(1)$ : Whenever  $a \sim x$  and  $a \sim \neg x$ , then  $a \sim \perp$ .

Although this rule is not probabilistically sound in general, it is sound for all probability functions  $p$  and thresholds  $t > 1/2$ . It is clearly implied by  $\mathbf{AND}$  (with  $\mathbf{RW}$ ) – but in the context of  $\mathbf{O}$  it is weaker than  $\mathbf{AND}$ .

*Observation 6.2.*  $\mathbf{R} = \mathbf{PU}\{\mathbf{RM}\} = \mathbf{OU}\{\mathbf{AND}, \mathbf{RM}\} = \mathbf{OU}\{\mathbf{PREF}(1), \mathbf{RM}\}$ .

*Observation 6.3.*  $\mathbf{S} = \mathbf{PU}\{\mathbf{ROR}\} = \mathbf{OU}\{\mathbf{AND}, \mathbf{ROR}\} = \mathbf{OU}\{\mathbf{PREF}(1), \mathbf{ROR}\} = (\mathbf{O} - \{\mathbf{WAND}\}) \cup \{\mathbf{PREF}(1), \mathbf{ROR}\}$ .

In each case the first equality is just the definition of the system, the second is immediate from Observation 4.1, and the remainder are not difficult to verify. We omit the details. We also clearly have:

*Observation 6.4.*  $\mathbf{S}$  is strictly stronger than  $\mathbf{R}$ , which is strictly stronger than the probabilistically sound family  $\mathbf{Q}$ .

In the light of Observations 6.2–6.4 we can say that the Horn rule  $\mathbf{PREF}(1)$  and non-Horn rule  $\mathbf{RM}$ , taken together, provide a bridge from probabilistic consequence to a strong form of qualitative consequence (the so-called ‘rational’ consequence relations), while  $\mathbf{PREF}(1)$  and non-Horn rule  $\mathbf{ROR}$  similarly provide a bridge to the very strong linear (or Stalnaker) consequence.

The difference between families  $\mathbf{R}$  and  $\mathbf{S}$  will assume further significance in part II of this paper, where we will define a hierarchy of rules that are probabilistically sound at various thresholds.  $\mathbf{S}$  implies all of them while  $\mathbf{R}$  implies none.

## Part II. Threshold-Sensitive Rules

In this part we extend the results of Part I by investigating families of threshold-sensitive probabilistically sound rules. We begin by examining threshold-sensitive (finite-premise) Horn rules that may be added to family  $\mathbf{O}$ . We obtain for each  $t$  in  $[0,1]$  a family  $\mathbf{O}(t)$  that is the strongest system of this kind that we know of. We then work with threshold-sensitive non-Horn rules, examining those that may be added to  $\mathbf{Q} = \mathbf{O} \cup \{\text{NR}\}$ , likewise obtaining a strongest known such system  $\mathbf{Q}(t)$ .

### 7. Threshold-Sensitive Horn Rules: the Preface Rules

The probabilistically sound closure principles we have treated thus far are probabilistically sound at each possible threshold level  $t$ . We now investigate additional rules that are probabilistically sound only when values of the threshold  $t$  are sufficiently high, or lie within some specific range of values. Among these are the ‘preface’ and ‘lottery’ rules studied by Hawthorne [8] and [9]. These rules are closely connected to the well-known lottery and preface paradoxes.

The *lottery paradox*, formulated by Kyburg [11], observes that if a fair lottery has a large number  $n$  of tickets, then, for each ticket, it is highly probable that it will not win, and thus rational to believe that it will not do so. At the same time, it is certain that some ticket among the  $n$  will win, and so rational to believe this too. But these  $n+1$  beliefs are inconsistent.

The *preface paradox*, formulated by Makinson [14], is similar in structure except that it makes no reference to probabilities. An author of a book making a large number  $n$  of assertions may have checked and rechecked each of them individually, and be confident of each that it is correct. But experience in these matters teaches that inevitably there will be errors somewhere among the  $n$  assertions, and the preface may acknowledge this. But, again, these  $n+1$  assertions are inconsistent.

Each of these epistemological puzzles suggests a series of rules that apply to some interesting families of consequence relations. In this section and the next we consider some (finite-premise) Horn rules motivated by the preface paradox. Then in section 9 we examine (finite-premise) non-Horn rules suggested by the lottery paradox.

To introduce the preface rules let’s return to the rule AND: conjunction in the conclusion. This tells us that whenever  $a \sim x_1$  and  $a \sim x_2$  then  $a \sim x_1 \wedge x_2$ . Applying it twice along with RW tells us:

Whenever  $a \sim x_1$  and  $a \sim x_2$  and  $a \sim \neg(x_1 \wedge x_2)$  then  $a \sim \perp$ .

Equivalently (again given RW):

Whenever  $a \sim x_1$  and  $a \sim x_2$  and  $a \sim \neg(x_1 \wedge x_2)$  then  $a \sim y$  for all  $y$ .

Since this rule follows from AND and RW, it is sound on the qualitative approach. Probabilistically it can fail when the threshold parameter  $t \leq 2/3$ , while it is easy to show that it holds whenever  $t > 2/3$ .

Generalizing from  $n = 2$  to arbitrary  $n \geq 1$ , we may formulate an infinite series of rules defined as follows:

PREF( $n$ ): Whenever  $a \sim x_1, \dots, a \sim x_n$  and  $a \sim \neg(x_1 \wedge \dots \wedge x_n)$ , then  $a \sim \perp$ .

We call these *preface rules*, because we may read  $x_1, \dots, x_n$  as the  $n$  assertions in the body of the book  $\neg(x_1 \wedge \dots \wedge x_n)$  as the statement in the preface, and  $a$  as the total background knowledge authorizing them. The rule PREF( $n$ ) says, in effect, that if our background beliefs give us good reason to believe each of the  $n$  assertions in the body of the text considered separately, and also provide good reason to believe that at least one of them fails, then they give us good reason to believe anything at all. The rule may thus be seen as corresponding to the injunction: “Don’t tolerate any belief that is in conflict with the conjunction of any sufficiently small set of propositions, each of which it individually supports”, where “small” is measured by the parameter  $n$ , conflict and support are both expressed in terms of  $\sim$ , and the conclusion  $a \sim \perp$  provides a criterion for not tolerating the belief  $a$ .

We now investigate the behavior of these rules, and of some others related to them, from the qualitative and quantitative perspectives. The remarks that follow draw from, reorganize, and add to those in Hawthorne [8] and [9]. To keep track of remarks, the reader should refer to the figure for Observation 7.3, which gathers them together.

Clearly PREF(2) is the rule mentioned above: whenever  $a \sim x_1$  and  $a \sim x_2$  and  $a \sim \neg(x_1 \wedge x_2)$  then  $a \sim \perp$ . PREF(1), mentioned near the end of section 6, says that whenever  $a \sim x$  and  $a \sim \neg x$  then  $a \sim \perp$ . For each  $n \geq 1$ , PREF( $n+1$ ) implies PREF( $n$ ): put  $x_{n+1} = x_n$  and apply RW.

The rules PREF( $n$ ) are all qualitatively sound: each is derivable from family **P** by applying AND  $n+1$  times and then RW. Distinguishing between them for different values of  $n$  is thus of no qualitative interest. On the other

hand, their quantitative status depends critically on the threshold chosen in the definition of probabilistic consequence.  $\text{PREF}(n)$  is probabilistically sound for sufficiently high values of  $t$  (given any fixed  $n > 0$ ) and likewise for sufficiently low values of  $n$  (given fixed  $t > 0.5$ ). Observation 7.2 will provide a precise statement and proof of this claim.

Alongside the series  $\text{PREF}(n)$ , consider a series of *classicality transforms*  $\text{CL-PREF}(n)$  for  $n \geq 1$ , where the snake in the last premise becomes a gate.

$\text{CL-PREF}(n)$ : Whenever  $a \vdash x_1, \dots, a \vdash x_n$  and  $a \vdash \neg(x_1 \wedge \dots \wedge x_n)$ , then  $a \vdash \perp$ .

Intuitively, each rule  $\text{CL-PREF}(n)$  tells us that if our background beliefs give us good reason to believe each of the assertions in the body of the text considered separately, but are at the same time logically inconsistent with those assertions considered together, then they give us good reason to believe anything at all.

Again, for each  $n \geq 1$ ,  $\text{CL-PREF}(n+1)$  implies  $\text{CL-PREF}(n)$ , by putting  $x_{n+1} = x_n$ . It is also clear that (since  $a \vdash y$  implies  $a \vdash y$ , by SUP),  $\text{PREF}(n)$  implies  $\text{CL-PREF}(n)$ . But we can show more.

*Observation 7.1.* Given the rules in family **O**,  $\text{PREF}(n)$  is equivalent to  $\text{CL-PREF}(n+1)$ .

PROOF. Clearly,  $\text{CL-PREF}(n+1)$  implies  $\text{PREF}(n)$ : simply take  $x_{n+1} = \neg(x_1 \wedge \dots \wedge x_n)$ . For the converse, suppose  $\text{PREF}(n)$  holds, and suppose  $a \vdash x_1, \dots, a \vdash x_n, a \vdash x_{n+1}$  and  $a \vdash \neg(x_1 \wedge \dots \wedge x_n \wedge x_{n+1})$ . We want to show using rules in **O** that  $a \vdash \perp$ . Since  $a \vdash x_{n+1}$  by supposition and  $a \wedge \neg a \vdash a$  by SUP, we have by WAND that  $a \vdash a \wedge x_{n+1}$ . And since  $a \vdash \neg(x_1 \wedge \dots \wedge x_n \wedge x_{n+1})$ , by classical logic  $a \wedge x_{n+1} \vdash \neg(x_1 \wedge \dots \wedge x_n)$ . So RW applied to  $a \vdash a \wedge x_{n+1}$  gives  $a \vdash \neg(x_1 \wedge \dots \wedge x_n)$ . Now apply  $\text{PREF}(n)$  to get  $a \vdash \perp$ , as desired. ■

It follows that each  $\text{CL-PREF}(n+1)$  is qualitatively sound, i.e. derivable from family **P**.  $\text{CL-PREF}(n+1)$  is also probabilistically sound for all probability functions  $p$  and all thresholds  $t > n/n+1$ , although for each  $t \leq n/n+1$  there is a  $p$  for which it fails (for a full proof, see Observation 7.2.) The limiting-case rule  $\text{CL-PREF}(1)$  follows from **O** alone; so  $\text{CL-PREF}(1)$  is both qualitatively sound and probabilistically sound for all thresholds  $t > 0$ .

In addition to the classicality transforms, we may consider the *failsafe transforms* of the preface rules. Following the terminology of section 3, these replace the last premise of each rule,  $a \vdash \neg(x_1 \wedge \dots \wedge x_n)$ , by its failsafe version

$a \approx \neg(x_1 \wedge \dots \wedge x_n)$ , i.e. by  $a \wedge (x_1 \wedge \dots \wedge x_n) \vdash \neg(x_1 \wedge \dots \wedge x_n)$ , giving rise to the series of rules:

FS-PREF( $n$ ): When  $a \vdash x_1, \dots, a \vdash x_n$  and  $a \approx \neg(x_1 \wedge \dots \wedge x_n)$ , then  $a \vdash \perp$ .

Using Observation 3.1, these rules may be written equivalently (modulo  $\mathbf{O}$ ):

FS-PREF( $n$ ): When  $a \vdash x_1, \dots, a \vdash x_n$  and  $a \wedge (x_1 \wedge \dots \wedge x_n) \vdash \perp$ , then  $a \vdash \perp$ .

The rule FS-PREF( $n$ ) says that if our background beliefs give us good reason to believe each of the assertions in the body of the text, and also makes it certain that at least one of them fails, then they give us good reason to believe anything at all. We recall from section 3 that this notion of one proposition ‘making another certain’ means, quantitatively, that the probability of the joint truth of the former with the negation of the latter is zero, or (in the limiting case) that the threshold chosen is itself zero. In qualitative terms, it means that this joint truth is not a seriously entertained possibility (more formally: is not satisfied in any world of the stoppered preferential model determining the consequence relation).

As before, for each  $n \geq 1$ , FS-PREF( $n+1$ ) implies FS-PREF( $n$ ), putting  $x_{n+1} = x_n$  and applying RW. Also, since  $a \vdash y$  implies  $a \wedge \neg y \vdash y$  which implies  $a \vdash y$  (given family  $\mathbf{O}$ ), we know that rule PREF( $n$ ) implies FS-PREF( $n$ ) which implies CL-PREF( $n$ ) (which is equivalent to PREF( $n-1$ )). Finally, it is not difficult to check that FS-PREF( $n+1$ ) behaves just like CL-PREF( $n+1$ ) with regard to probabilistic soundness:

*Observation 7.2.* (Hawthorne [8], [9]). For  $n \geq 1$ , FS-PREF( $n+1$ ) and CL-PREF( $n+1$ ) (and also PREF( $n$ )) are probabilistically sound for all probability functions  $p$  and all thresholds  $t > n/n+1$ , although for each  $t \leq n/n+1$  there are  $p$  for which each rule fails.

The verification is in the appendix.

The limiting-case rule FS-PREF(1) follows from  $\mathbf{O}$  alone; so FS-PREF(1) is both qualitatively sound and probabilistically sound for all thresholds  $t > 0$ .

*Observation 7.3.* Putting all this together, we have the configuration of implications modulo family  $\mathbf{O}$  shown by Figure 4.

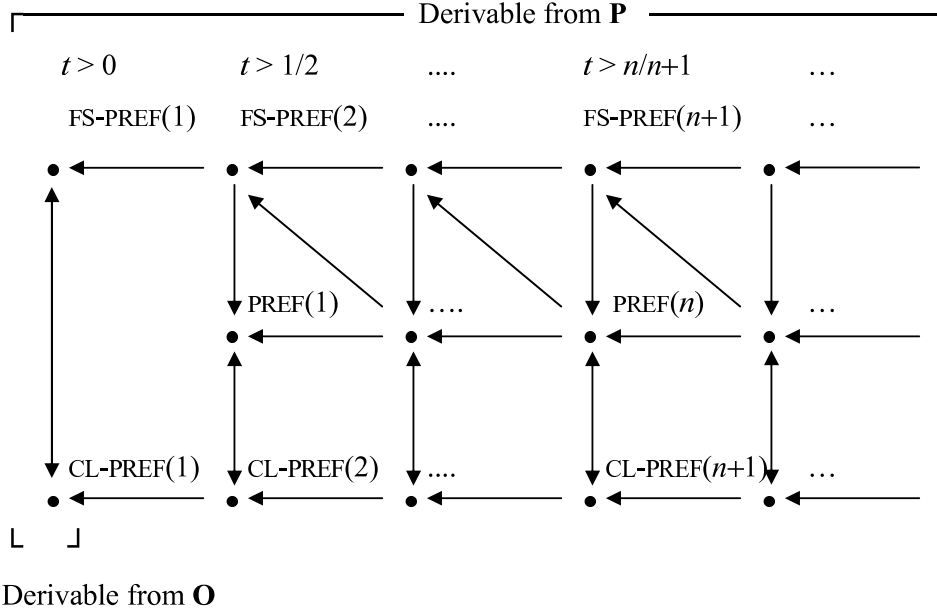


Figure 4. Figure for Observation 7.3

Qualitatively, all these rules are derivable from  $\mathbf{P}$  and so have no individual interest from a qualitative point of view. Probabilistically,  $\text{PREF}(n)$ ,  $\text{CL-PREF}(n+1)$ , and  $\text{FS-PREF}(n+1)$  are each sound for threshold  $t$  iff  $t > n/n+1$ .  $\text{FS-PREF}(1)$  and  $\text{CL-PREF}(1)$  are both derivable from  $\mathbf{O}$ . It follows easily that the  $\text{FS-PREF}$  rules have a distinguishing property.

*Corollary 7.4.* Let  $t \in (1/2, 1)$ . Choose  $n$  to be the largest positive integer such that  $t > n/n+1$  (i.e. the largest  $n$  such that  $n < t/1 - t$ ). Then  $\text{FS-PREF}(n+1)$  is strictly the strongest rule in the array of the Figure for Observation 7.3 that's sound for probabilistic consequence with threshold  $t$ .

The verification is in the appendix.

The  $\text{PREF}$  rules are sound only for thresholds  $t > 1/2$ . Now consider a family of weaker consequence relations, a family where  $a \vdash x$  says, “if  $a$  holds, then  $x$  is somewhat plausible.” Consequence relations of this sort correspond to conditional probabilities for thresholds  $t \leq 1/2$ . The following rule is a sort of continuation of the  $\text{FS-PREF}$  rules to thresholds  $t > 1/n+1$

for  $n \geq 1$ . But they have a somewhat different character than the PREF rules. Let's call them the PLAUS (for 'plausibility') rules:

$$\text{FS-PLAUS}(n+1): \text{ When } a \vdash x_1, \dots, a \vdash x_{n+1} \text{ and } a \approx \neg(x_1 \wedge x_2), \\ a \approx \neg(x_1 \wedge x_3), \dots, a \approx \neg(x_n \wedge x_{n+1}), \text{ then } a \vdash \perp.$$

Notice that FS-PLAUS(2) is the same rule as FS-PREF(2). Both are probabilistically sound at all thresholds  $t > 1/2$ . And just as each FS-PREF( $n+1$ ) rule is sound for all  $t > n/n+1$ , it turns out that each FS-PLAUS( $n+1$ ) is probabilistically sound for all thresholds  $t > 1/n+1$ . Thus each FS-PLAUS( $n+1$ ) is probabilistically sound for sufficiently high values of  $t$  (given any fixed  $n > 0$ ) and likewise for sufficiently high values of  $n$  (given fixed  $t > 0$ ).

*Observation 7.5.* (Hawthorne [8]). For  $n \geq 1$ , FS-PLAUS( $n+1$ ) is probabilistically sound for all probability functions  $p$  and all thresholds  $t > 1/n+1$ , although for each  $t \leq 1/n+1$  there are  $p$  for which it fails.

The verification is in the appendix.

Clearly, for each  $n \geq 1$ , FS-PLAUS( $n+1$ ) implies FS-PLAUS( $n+2$ ). And we've already seen that FS-PREF( $n+2$ ) implies FS-PREF( $n+1$ ), and that FS-PREF(2) is the same rule as FS-PLAUS( $n+2$ ). Thus we have the following hierarchy of rules for thresholds  $t$ .

*Observation 7.6.* Putting all of the observations relating FS-PREF and FS-PLAUS together, we have the configuration of implications modulo family **O** shown by Figure 5.

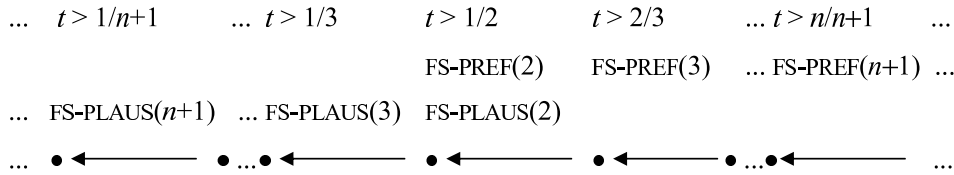


Figure 5. Figure for Observation 7.6

Thus the FS-PREF and FS-PLAUS rules, taken together and ordered by strength, form an infinite sequence, both down and up, with the common rule FS-PREF(2) = FS-PLAUS(2) as central point. In the next section we further generalize these rules.

## 8. More General Threshold-Sensitive Preface and Plausibility Rules

Each FS-PREF( $n+1$ ) rule (for each value of  $n \geq 1$ ) is a special cases of a yet more general preface-like rule that is probabilistically sound for an appropriate threshold level  $t$ . To introduce these rules intuitively, consider the failsafe version of the usual preface rule, but contraposed, so that we suppose the background belief  $a$  to be both coherent (i.e.  $a \not\vdash \perp$ ) and to imply that each of  $n+1$  pages is error free (i.e.  $a \vdash x_1, \dots, a \vdash x_{n+1}$ ). Then (for  $n$  small enough relative to threshold  $t$ ) the agent should not be certain that at least one page has an error – that is, we should have  $a \not\vdash \neg(x_1 \wedge \dots \wedge x_{n+1})$ , or equivalently,  $a \wedge (x_1 \wedge \dots \wedge x_{n+1}) \not\vdash \perp$ .

Now consider a similar case, but where background belief  $a$  is not itself sufficient to support the claim that page  $i$  is error free, so that  $a \vdash x_i$  fails to hold. Nevertheless, suppose the agent has a chain argument that begins with background  $a$  and supports a cumulative series of intermediate conclusions  $x_{i,1}, x_{i,2}, \dots, x_{i,k_i}$  (about the reliability of the process of fact-checking page  $i$ ) that ultimately jointly support the claim that page  $i$  is error free (expressed by the statement  $x_{i,k_i}$ ):  $a \vdash x_{i,1}, a \wedge x_{i,1} \vdash x_{i,2}, \dots, a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \vdash x_{i,k_i}$ . For example,  $x_{i,1}$  may say that the proof reader is conscientious,  $x_{i,2}$  that she checked the references on page  $i$  carefully,  $x_{i,3}$  that she used reliable sources to check the references on page  $i$ , ..., etc., where each  $x_{i,j}$  may draw on whatever it needs from the conjunction  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,j-1}$  for support.

Furthermore, the “error free” conclusions for many of the pages may rely on such chain arguments of various lengths – perhaps chains consisting of quite different premises for each page – premises specific to the kind of information that specific page contains.

In such a case, a more general version of the preface rule, the CA-PREF( $t$ ) rule applies. It says that if the number of claims about pages  $n+1$  is small enough and the lengths of each of the argument chains is short enough (so that in combination  $n/n+1 < (\sum_{i=1}^{n+1} t^{k_i})/n+1$  – i.e.  $n/n+1$  is smaller than the average value of  $t^{k_i}$ ), then  $a$  must be incompatible with the conjunction of all of the claims (premises and conclusions) involved. If, however, the number of conclusion claims,  $n+1$ , and the lengths of the argument chains for them, the  $k_i$ , are jointly big enough relative to the threshold  $t$ , then  $a$  may be perfectly compatible with the conjunction of all the claims involved.

In formulating the appropriate generalizations of the preface rules, it will prove convenient to index the new rules by the thresholds  $t$  at which they are sound. Here is what the new rules look like.



CA-PREF( $t$ ): For  $n \geq 1$ , for any  $n+1$  integers  $k_i \geq 1$  ( $1 \leq i \leq n+1$ ) such that  $n < \sum_{i=1}^{n+1} t^{k_i}$  (i.e., such that  $n/n+1 < (\sum_{i=1}^{n+1} t^{k_i})/n+1$ , the average value of the  $t^{k_i}$ ):

when for all  $i$  such that  $1 \leq i \leq n+1$ ,

$a \vdash x_{i,1}$ ,  $a \wedge x_{i,1} \vdash x_{i,2}$ , ...,  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \vdash x_{i,k_i}$ , and

$\bigwedge_{i=1}^{n+1} (a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i}) \vdash \perp$ ,

then  $a \vdash \perp$ .

The letters ‘CA’ stand for ‘Chain Argument’. This new rule is a Chain Argument (failsafe version) Preface Rule. FS-PREF( $n+1$ ) is a special case of this new rule where the length of each argument chain from  $a$  to the conclusion  $x_{i,k_i}$  is 1 (i.e. where each of the  $k_i = 1$ ). In general, for each  $t > n/n+1$ , the CA-PREF( $t$ ) rule reduces to the FS-PREF( $n+1$ ) rule whenever the length  $k_i$  of each chain argument is 1 and there are precisely  $n+1$  conclusion propositions involved.

Whereas it makes good sense to index the FS-PREF rules by the number,  $n+1$ , of conclusion propositions involved (because the appropriate threshold can be directly recovered from it), the CA-PREF rules depend on a complex combination of the number of conclusion propositions,  $n+1$ , the argument chain lengths, and the threshold. Since our goal in this paper is to characterize the relationship between qualitative rules and quantitative rules, and since this relationship is most easily seen by indexing the rules in a way closely tied to the quantitative threshold, it makes good sense to index the CA-PREF rules by the relevant threshold  $t$ .

*Observation 8.1.* CA-PREF( $t$ ) is probabilistically sound for all thresholds  $t$ .

The verification is in the appendix. And it is also easy to check that each CA-PREF( $t$ ) rule is derivable from  $\mathbf{P} = \mathbf{OU}(\text{AND})$ .

Notice that although this rule does not assume that  $t > 1/2$ , it only really applies (i.e. it only has a non-vacuous antecedent) when  $t > 1/2$ ; because, if  $t \leq 1/2$ , we’d have  $n < \sum_{i=1}^{n+1} t^{k_i} \leq (n+1)(1/2)$ , so  $2n < (n+1)$ , so  $n < 1$ , which isn’t possible. This suggests that there should be a series of additional rules, like the FS-PLAUS rules, that apply (and are effective) for thresholds  $t \leq 1/2$ . Here is the more general counterpart of the schema for the FS-PLAUS rule.

CA-PLAUS( $t$ ): For  $n \geq 1$ , for any  $n+1$  integers  $k_i \geq 1$  ( $1 \leq i \leq n+1$ ) such that  $1 < \sum_{i=1}^{n+1} t^{k_i}$  (i.e., such that  $1/n+1 < (\sum_{i=1}^{n+1} t^{k_i})/n+1$ , the average value of the  $t^{k_i}$ ):

when for all  $i$  and  $j$  such that  $1 \leq i < j \leq n+1$ ,  
 $(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i}) \wedge (a \wedge x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,k_j}) \vdash \perp$ ,  
and for all  $i$  such that  $1 \leq i \leq n+1$ ,  
 $a \vdash x_{i,1}$ ,  $a \wedge x_{i,1} \vdash x_{i,2}$ , ...,  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \vdash x_{i,k_i}$ ,  
then  $a \vdash \perp$ .

Notice that the CA-PLAUS( $t$ ) rule reduces to the FS-PLAUS( $n+1$ ) rule(s) for  $n/n+1 < t$  whenever the length  $k_i$  of each chain argument is 1 and there are precisely  $n+1$  conclusion propositions involved. Also notice that when  $t = 0$ , this rule is ineffective, since in that case  $\sum_{i=1}^{n+1} t^{k_i} = 0$  but the antecedent of the rule requires that  $\sum_{i=1}^{n+1} t^{k_i} > 1$ . This rule is probabilistically sound for each threshold  $t \geq 0$ , as the next observation shows.

*Observation 8.2.* CA-PLAUS( $t$ ) is probabilistically sound for all thresholds  $t$ .

The verification is in the appendix. And it is also easy to check that each CA-PLAUS( $t$ ) rule is derivable from  $\mathbf{P} = \mathbf{OU}(\text{AND})$ .

Given the rules of  $\mathbf{O}$ , for each  $t > t'$ , the rule CA-PREF( $t$ ) implies the rule CA-PREF( $t'$ ), but not *vice versa*, and the rule CA-PLAUS( $t$ ) implies the rule CA-PLAUS( $t'$ ), but not *vice versa*. This fact suggests the following definition.

*Definition 8.1.* For each possible threshold  $t$  such that  $0 \leq t \leq 1$ , define the family  $\mathbf{O}(t)$  of closure conditions on a consequence relation by putting  $\mathbf{O}(0) = \mathbf{O}$ , and for all  $t > 0$ ,  $\mathbf{O}(t) = \mathbf{OU}\{\text{CA-PREF}(t), \text{CA-PLAUS}(t)\}$ .

For each  $t > t'$ , the family  $\mathbf{O}(t)$  implies the family  $\mathbf{O}(t')$ , but not *vice versa*. Recalling that  $\mathbf{P} = \mathbf{OU}\{\text{AND}\}$ , so that  $\mathbf{P}$  implies each  $\mathbf{O}(t)$ , we have  $\mathbf{P}$  at the top of this hierarchy. In terms of families of consequence relations, as  $t$  increases, the hierarchy of  $\mathbf{O}(t)$  rules represents a hierarchy of ever stronger logics, each capturing a proper subset of the set of consequence relations in families below it.

## 9. Threshold-Sensitive non-Horn Rules: the Lottery and Other Sound Rules

We now draw attention to a hierarchy of threshold-dependent non-Horn rules that follow from the very strong qualitative ‘linear’ (alias Stalnaker) family

**S**, although not from the less strong ‘rational’ system **R**, defined in section 6. We call these new rules ‘Lottery Rules’ because of their connection to a version of the lottery paradox. Just as the family **P** implies rules like  $\text{PREF}(n)$ ,  $\text{FS-PREF}(n+1)$ , and  $\text{FS-PLAUS}(n+1)$ , which are probabilistically sound at all thresholds  $t > n/n+1$  (but unsound for all lower thresholds), this series of non-Horn rules are probabilistically sound at all thresholds  $t \leq n+1/n+2$  (but unsound at all higher thresholds). **R** does not imply these additional rules, but **S** does. Here is the form these rules take.

$\text{FS-LOTT}(n)$ : Whenever for  $n \geq 2$  distinct formulae,  $x_1, \dots, x_n$ ,  $a \approx \neg(x_1 \wedge x_2)$ ,  $a \approx \neg(x_1 \wedge x_3)$ , ...,  $a \approx \neg(x_{n-1} \wedge x_n)$ , then  $a \vdash \neg x_1$  or ... or  $a \vdash \neg x_n$ .

Notice that for each  $n$  the rule  $\text{FS-LOTT}(n)$  implies the rule  $\text{FS-LOTT}(n+1)$ .

Rule  $\text{FS-LOTT}(n)$  says that if background beliefs  $a$  make it certain that no two of a list of  $n$  distinct the propositions are both true, then for at least one of these propositions, the background beliefs  $a$  must provide good reason to believe that proposition is false. The connection to the lottery paradox can be seen by taking ‘ $a$ ’ to describe a lottery for which no two tickets are permitted to win, and by reading each ‘ $x_i$ ’ as saying that “ticket  $i$  will win”. The fail-safe relations  $a \approx \neg(x_j \wedge x_k)$  (i.e.  $a \wedge (x_j \wedge x_k) \vdash \neg(x_j \wedge x_k)$ ) express the certainty that tickets  $j$  and  $k$  cannot both win.

Notice that this rule makes no supposition that “one of the tickets will win” – i.e. it does not suppose ‘ $a \vdash (x_1 \vee \dots \vee x_n)$ ’ – and there is no supposition that each ticket has the same chance of winning. (If all tickets have the same chance of winning, then we should also have ‘ $a \vdash \neg x_i$  iff  $a \vdash \neg x_j$ ’ for each  $x_i$  and  $x_j$ , and the consequent of the rule would imply the conjunction ‘ $a \vdash \neg x_1$  and ... and  $a \vdash \neg x_n$ ’.) The idea, then, is that if the lottery consists of a *very large* number of tickets, and if our background beliefs  $a$  make it certain that at most one ticket can win (and if our threshold for belief isn’t too near 1), then  $a$  must support the belief that ticket  $j$  will lose, for at least one of the tickets  $j$ .

*Observation 9.1.* **S** implies each of the  $\text{FS-PREF}$  rules. Indeed the rules  $\mathbf{OU}\{\text{ROR}\}$  implies them.

**PROOF.** Given  $a \approx \neg(x_1 \wedge x_2)$  (i.e.  $a \wedge (x_1 \wedge x_2) \vdash \neg(x_1 \wedge x_2)$ ), we have (from **O**)  $a \vdash \neg(x_1 \wedge x_2)$ , so  $a \vdash (\neg x_1 \vee \neg x_2)$ , so by **ROR** we have ‘ $a \vdash \neg x_1$  or  $a \vdash \neg x_2$ ’. The rest is obvious. ■

Whereas ROR is not probabilistically sound at any threshold level, each rule FS-LOTT( $n+1$ ), for  $n \geq 1$ , is probabilistically sound for sufficiently low values of  $t$  (given any fixed  $n > 0$ ), and likewise for sufficiently high values of  $n$  (given fixed  $t > 0$ ).

*Observation 9.2.* (Hawthorne [8], [9]). For  $n \geq 1$ , FS-LOTT( $n+1$ ) is probabilistically sound for all probability functions  $p$  and all thresholds  $t \leq n/n+1$ , although for each  $t > n/n+1$  there are  $p$  for which it fails.

This is verified in the appendix.

Since FS-LOTT( $n+1$ ) is probabilistically sound precisely when threshold  $t \leq n/n+1$  while FS-PREF( $m+1$ ) is probabilistically sound precisely when  $t > m/m+1$ , these two rules are sound together precisely when the threshold  $t$  is between these two bounds – i.e., precisely when  $n/n+1 \geq t > m/m+1$ , for  $n > m \geq 1$ . Clearly the narrowest such bounds on  $t$  occur for  $n/n+1 \geq t > n-1/n$ , for values of  $n \geq 2$  – i.e. for pairs of rules FS-LOTT( $n+1$ ) and FS-PREF( $n$ ), for  $n \geq 2$ .

The FS-LOTT rules are only “effective” down to an upper bound of  $1/2$  on the value of the threshold  $t$ . However, there are additional probabilistically sound rules for thresholds  $t$  bounded above smaller fractions – e.g., by  $1/3$ , by  $1/4$ , and more generally by  $1/n$ . The FS-LOTT rules will remain sound at these smaller bounds, but are subsumed by (i.e. implied by) the rules appropriate to these bounds. These new rules, associated with bounds  $1/n$ , are appropriate for the families of weaker consequence relations we identified earlier, those where  $a \sim x$  says, “if  $a$  holds, then  $x$  is somewhat plausible”. These are the same relations we associated with the FS-PLAUS rules. Consequence relations of this sort correspond to conditional probabilities for thresholds  $t \leq 1/2$ . The following rule is a sort of continuation of the FS-LOTT rules to thresholds  $t \leq 1/n$  for  $n \geq 2$ . Let’s call them the POSS (for ‘possibility’) rules:

FS-POSS( $n$ ): Whenever for  $n \geq 2$  distinct formulae  $x_1, \dots, x_n$ ,  
 $a \not\approx \neg(x_1 \wedge \dots \wedge x_n)$ , then  $a \sim \neg x_1$  or ... or  $a \sim \neg x_n$ .

Equivalently, we can express this rule as follows:

FS-POSS( $n$ ): Whenever for  $n \geq 2$  distinct formulae,  $x_1, \dots, x_n$ ,  
 $a \wedge (x_1 \wedge \dots \wedge x_n) \sim \perp$ , then  $a \sim \neg x_1$  or ... or  $a \sim \neg x_n$ .

Notice that FS-POSS(2) is the same rule as FS-LOTT(2). Both are probabilistically sound at all thresholds  $t \leq 1/2$ . Furthermore, clearly, each rule FS-POSS( $n+1$ ) implies the rule FS-POSS( $n$ ) (and every FS-POSS( $m$ ) rule for  $m \leq n$ ), and all rules FS-LOTT( $n$ ) as well. Furthermore, just as each FS-LOTT( $n+1$ ) rule is sound for all  $t \leq n/n+1$ , it turns out that each FS-POSS( $n+1$ ) is probabilistically sound for all thresholds  $t \leq 1/n+1$ .

Thus each rule FS-POSS( $n+1$ ), is probabilistically sound for sufficiently low values of  $t$  (given any fixed  $n > 0$ ) and likewise for sufficiently low values of  $n$  (given fixed  $t \leq 1/2$ ).

*Observation 9.3.* (Hawthorne [8]). For  $n \geq 1$ , FS-POSS( $n+1$ ) is probabilistically sound for all probability functions  $p$  and all thresholds  $t \leq 1/n+1$ , although for each  $t > 1/n+1$  there are  $p$  for which it fails.

The verification is in the appendix.

It is easy to see that the qualitative system **S** implies the FS-POSS rules.

*Observation 9.4.* **S** implies each of the FS-POSS rules. Indeed the rules  $\mathbf{O} \cup \{\mathbf{ROR}\}$  implies them.

PROOF. Given  $a \approx \neg(x_1 \wedge \dots \wedge x_n)$ , from **O** we have  $a \vdash \neg(x_1 \wedge \dots \wedge x_n)$ , so  $a \vdash (\neg x_1 \vee \dots \vee \neg x_n)$ , so by repeated applications of ROR we have ‘ $a \vdash \neg x_1$  or  $a \vdash \neg x_2$ , or ... or  $a \vdash \neg x_n$ . ■

Putting the two new classes of rules described in this section together with the related threshold-sensitive Horn rules discussed in section 7, we get the following picture. For each threshold  $t$  between 0 and 1, the rules of **Q** are probabilistically sound. In addition, for each  $n \geq 2$ , whenever  $n/n+1 \geq t > n-1/n$ , the additional threshold-sensitive rules FS-LOTT( $n+1$ ) and FS-PREF( $n$ ) are probabilistically sound. And for each  $n \geq 2$ , whenever  $1/n \geq t > 1/n+1$ , the threshold-sensitive rules FS-POSS( $n$ ) and FS-PLAUS( $n+1$ ) are probabilistically sound. Thus, given the implications among these rules we have the following hierarchy of rules for thresholds  $t$ .

*Observation 9.5.* Putting all of the observations relating FS-PREF, FS-PLAUS, FS-LOTT, and FS-POSS together, we have the configuration of implications modulo family **Q** given by Figure 6.

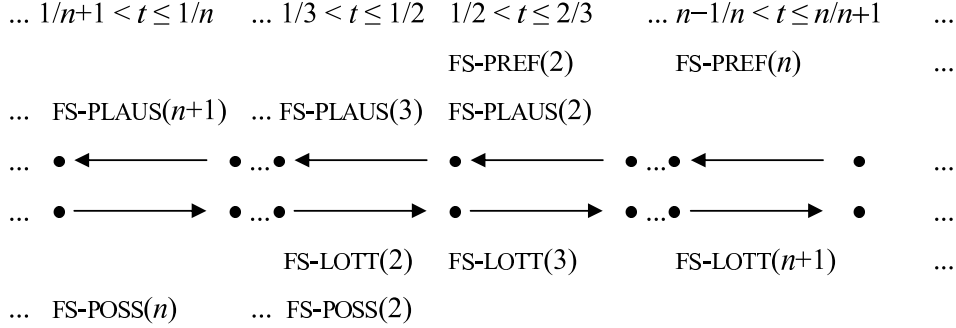


Figure 6. Figure for Observation 9.5

This very nearly covers the strongest class of probabilistically sound rules we know of. But there is one additional wrinkle – one further generalization. Recall that the FS-PREF and FS-PLAUS rules were each subject to generalizations that involved chain arguments – giving rise to the probabilistically sound CA-PREF and CA-PLAUS rules. The two classes of rules introduced in the present section are also generalizable to probabilistically sound chain argument versions, CA-LOTT and CA-POSS, which have the rules FS-LOTT and FS-POSS, respectively, as special cases.

Here is the chain argument version of the FS-LOTT rule.

CA-LOTT( $t$ ): For  $n \geq 1$ , for any  $n+1$  integers  $k_i \geq 1$  ( $1 \leq i \leq n+1$ ) such that  $1 \leq \sum_{i=1}^{n+1} (1-t)^{k_i}$  (i.e., such that  $1/n+1 \leq (\sum_{i=1}^{n+1} (1-t)^{k_i})/n+1$ , the average value of the  $(1-t)^{k_i}$ ):

when for all  $i$  such that  $1 \leq i \leq n+1$ ,

$a \not\vdash \neg x_{i,1}, a \wedge x_{i,1} \not\vdash \neg x_{i,2}, \dots, a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \not\vdash \neg x_{i,k_i}$ ,

then, for some  $i$  and  $j$  such that  $1 \leq i < j \leq n+1$ ,

$(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \wedge (a \wedge x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,k_j-1} \wedge x_{j,k_j}) \not\vdash \perp$ .

This Chain Argument Lottery rule holds vacuously when the threshold  $t = 1$ . But the rule has substance for thresholds  $t$  less than 1 provided that  $n$  is large enough and the  $k_i$  are small enough. Also notice that when this rule is probabilistically sound for a given  $t$ , set of  $k_i$ , and  $n$ , then (keeping  $t$  fixed) for all larger values of  $n$  and all smaller values of the  $k_i$ , the rule must continue to hold. When each  $k_i = 1$ , this rule is just the FS-LOTT( $n+1$ ) rule for the smallest  $n$  such that  $n/n+1 \geq t$ .

*Observation 9.6.* CA-LOTT( $t$ ) is probabilistically sound for all thresholds  $t$ .

The verification is in the appendix. It is also easy to show that the qualitative system  $\mathbf{S}$  implies the CA-LOTT( $t$ ) rules.

The chain argument version of the FS-POSS rule is as follows.

CA-POSS( $t$ ): For all  $n \geq 1$  and any  $n+1$  integers  $k_i \geq 1$  ( $1 \leq i \leq n+1$ ) such that  $n \leq \sum_{i=1}^{n+1} (1-t)^{k_i}$  (i.e., such that  $n/n+1 \leq (\sum_{i=1}^{n+1} (1-t)^{k_i})/n+1$ ):

when for all  $i$  such that  $1 \leq i \leq n+1$ ,

$a \not\vdash \neg x_{i,1}, a \wedge x_{i,1} \not\vdash \neg x_{i,2}, \dots, a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \not\vdash \neg x_{i,k_i}$ ,

then  $\bigwedge_{i=1}^{n+1} (a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \not\vdash \perp$ .

Notice that for this rule to take effect  $t$  cannot be greater than  $1/2$  (because, if  $t > 1/2$ , then we'd have  $(1-t) < 1/2$ , so  $n \leq \sum_{i=1}^{n+1} (1-t)^{k_i} < (n+1)(1/2)$ , so  $2n < (n+1)$ , so  $n < 1$ ). But  $t \geq 0$  can have any value less than or equal to  $1/2$  (provided that  $n$  and the  $k_i$  are small enough). Also notice that when this rule is sound for a given  $t$ , set of  $k_i$ , and  $n$ , then (keeping  $t$  fixed) for all smaller values of  $n$  and the  $k_i$ , the rule continues to hold. When each  $k_i = 1$  this rule is just the FS-POSS( $n+1$ ) rule.

*Observation 9.7.* CA-POSS( $t$ ) is probabilistically sound for all thresholds  $t$ .

The verification is in the appendix. It is also easy to show that the qualitative system  $\mathbf{S}$  implies the CA-POSS( $t$ ) rules.

The strongest families of threshold-sensitive probabilistically sound rules we've investigated in this paper consist of the rules in  $\mathbf{Q} = \mathbf{O} \cup \{\text{NR}\}$  together with the rules CA-PREF( $t$ ), CA-PLAUS( $t$ ) (in  $\mathbf{O}(t)$ ), plus the rules CA-LOTT( $t$ ), CA-POSS( $t$ ). This suggests the following definition.

*Definition 9.1.* For each  $t \in [0,1]$ , define the family  $\mathbf{Q}(t)$  of closure conditions on a consequence relation by putting  $\mathbf{Q}(t) = \mathbf{O}(t) \cup \{\text{NR}, \text{CA-LOTT}(t), \text{CA-POSS}(t)\} = \mathbf{O} \cup \{\text{CA-PREF}(t), \text{CA-PLAUS}(t), \text{NR}, \text{CA-LOTT}(t), \text{CA-POSS}(t)\}$ .

The threshold-sensitive rules associated with chain arguments do not conflict in any way. Each is formulated so as to remain sound for all thresholds  $t$  in the interval  $[0,1]$ . However, among these rules, when  $t > 1/2$ , only the rules CA-PREF( $t$ ), CA-PLAUS( $t$ ), and CA-LOTT( $t$ ) apply. The rule CA-POSS( $t$ ) remains consistent, but it fails to apply when  $t > 1/2$  because its antecedent condition,  $n \leq \sum_{i=1}^{n+1} (1-t)^{k_i}$ , must be vacuous in that case

(because, when  $t > 1/2$  we have  $(1 - t) < 1/2$ , so  $n \leq \sum_{i=1}^{n+1} (1 - t)^{k_i} < (n+1)(1/2)$ , so  $2n < (n+1)$ , so  $n < 1$ ). Similarly, when  $t \leq 1/2$ , only the rules CA-PLAUS( $t$ ), and CA-LOTT( $t$ ), and CA-POSS( $t$ ) apply. The rule CA-PREF( $t$ ) remains consistent, but it fails to apply when  $t \leq 1/2$  because its antecedent condition,  $n < \sum_{i=1}^{n+1} t^{k_i}$ , must be vacuous (because, when  $t \leq 1/2$  we have  $n < \sum_{i=1}^{n+1} t^{k_i} \leq (n+1)(1/2)$ , so  $2n < (n+1)$ , so  $n < 1$ ).

The following observation sums up our discussion of the families of threshold-sensitive rules.

*Observation 9.8.* For each  $t \in [0,1]$ , the family of rules  $\mathbf{Q}(t)$  is probabilistically sound.

$\mathbf{O}(t)$  is the strongest family of threshold-sensitive probabilistically sound *Horn rules* we have found. But when we consider all probabilistically sound threshold-sensitive rules, Horn and non-Horn,  $\mathbf{Q}(t)$  emerges as the strongest family of such rules we have been able to identify.

## 10. Open Questions

We gather together some open questions that were mentioned in the text. Perhaps the most basic is that of representation:

*Question 1.* Can a representation theorem be established for family  $\mathbf{O}$  plus negation rationality plus the ‘Archimedean rule’ of section 5?

Our conjecture is negative.

The next question concerns the completeness of family  $\mathbf{O}$  with respect to finite-premise Horn rules.

*Question 2.* Is every finite-premise Horn rule that is satisfied by all probabilistically defined consequence relations derivable from family  $\mathbf{O}$ ?

We know from Observation 4.3 that this holds for one-premise Horn rules, and indeed for all finite-premise Horn rules with pairwise inconsistent premise antecedents. But we also know from Corollary 5.2 that it can fail for countable-premise Horn rules. Moreover, by Observation 5.1, if the answer to this question is positive, it cannot be established via a representation theorem. Our conjecture for Question 2 is nevertheless positive.

The question of completeness also arises when we consider threshold-sensitive probabilistic consequence relations.

*Question 3.* Consider the set of all consequence relations defined probabilistically with a fixed threshold value  $t \in (0, 1)$ . Is every finite-premise Horn rule that is satisfied by all of these consequence relations derivable



from those in the family  $\mathbf{O}(t)$ ? In other words, is  $\mathbf{O}(t)$  a complete axiomatization of the finite-length Horn rules satisfied by the threshold  $t$  probabilistic consequence relations, for each threshold  $t$ ? And is  $\mathbf{Q}(t)$  a complete axiomatization of *all* (Horn and non-Horn) finite-length rules satisfied by probabilistic consequence relations at threshold  $t$ ?

### Appendix: Verifications

This appendix contains a number of proofs omitted from the body of the text. They are of two kinds. Some (particularly from Part II) are rather long but unexciting verifications whose presence in the main text would have impeded the reader's progress. Others (particularly from Part I) include proofs of facts used in this paper that were established in different contexts and sometimes in different ways by Hawthorne [8] or Bochman [4].

*Observation 2.3.* (Hawthorne [8]). Probabilistic consequences satisfy all the conditions in family  $\mathbf{O}$ .

PROOF. REFLEX, RW, and LCE are immediate. For the others, the details are as follows. Let  $p$  be any probability function and  $t \in [0,1]$ . Let  $\vdash$  be the consequence relation defined modulo  $p$  and  $t$ .

For WOR: Suppose  $a \wedge b \vdash x$  and  $a \wedge \neg b \vdash x$ . We need to show  $a \vdash x$ . If  $p(a) = 0$  this is immediate. So suppose  $p(a) > 0$ .

Case 1: Suppose  $p(a \wedge b) = 0$ . Then  $0 < p(a) = p(a \wedge \neg b)$ , so since by supposition  $a \wedge \neg b \vdash x$  we have  $t \leq p(a \wedge \neg b \wedge x)/p(a \wedge \neg b) = p(a \wedge \neg b \wedge x)/p(a) \leq p(a \wedge x)/p(a)$  so  $a \vdash x$ .

Case 2: Suppose  $p(a \wedge \neg b) = 0$ . Then similarly  $0 < p(a) = p(a \wedge b)$ ; so since by supposition  $a \wedge b \vdash x$  we have  $t \leq p(a \wedge b \wedge x)/p(a \wedge b) = p(a \wedge b \wedge x)/p(a) \leq p(a \wedge x)/p(a)$  so  $a \vdash x$ .

Case 3: Suppose finally  $p(a \wedge b) > 0$  and  $p(a \wedge \neg b) > 0$ . Then by the suppositions  $p(a \wedge b \wedge x) \geq p(a \wedge b) \cdot t$  and  $p(a \wedge \neg b \wedge x) \geq p(a \wedge \neg b) \cdot t$ , so  $p(a \wedge x) = p(a \wedge x \wedge b) + p(a \wedge x \wedge \neg b) \geq [p(a \wedge b) + p(a \wedge \neg b)] \cdot t = p(a) \cdot t$ , so  $p(a \wedge x)/p(a) \geq t$ , that is,  $a \vdash x$ .

For VCM: Suppose  $a \vdash x \wedge y$ . We need to show  $a \wedge x \vdash y$ . If  $p(a \wedge x) = 0$ , then  $a \wedge x \vdash y$  and we are done. So suppose  $p(a \wedge x) > 0$ . Then  $p(a) > 0$ , so the initial supposition gives us  $p(a \wedge x \wedge y) \geq p(a) \cdot t \geq p(a \wedge x) \cdot t$ , so  $p(a \wedge x \wedge y)/p(a \wedge x) \geq t$ , i.e.  $a \wedge x \vdash y$  as desired.

For WAND: Suppose  $a \vdash x$  and  $a \wedge \neg y \vdash y$ . We need to show  $a \vdash x \wedge y$ . If  $p(a) = 0$ , then  $a \vdash x \wedge y$  and we are done. So suppose  $p(a) > 0$ .

Case 1: Suppose  $p(a \wedge \neg y) = 0$ . Then  $p(a \wedge x \wedge \neg y) = 0$ , so  $p(a \wedge x) = p(a \wedge x \wedge y) + p(a \wedge x \wedge \neg y) = p(a \wedge x \wedge y)$ . So, since  $a \vdash x$ , we have  $p(a \wedge x \wedge y) = p(a \wedge x) \geq p(a) \cdot t$ . Thus,  $p(a \wedge x \wedge y)/p(a) \geq t$ , that is,  $a \vdash x \wedge y$ .

Case 2: Suppose  $p(a \wedge \neg y) > 0$ . Then, since  $a \wedge \neg y \vdash y$ , we have  $0 = p(a \wedge \neg y \wedge y)/p(a \wedge \neg y) \geq t$ , so  $t = 0$ , and so  $p(a \wedge x \wedge y)/p(a) \geq t = 0$ , that is,  $a \vdash x \wedge y$ . ■

*Observation 3.2.* Every rule in  $\text{FS}(\mathbf{P})$  is derivable from family  $\mathbf{O}$ . But not conversely: WOR is not derivable from  $\text{FS}(\mathbf{P})$ . Indeed, WOR is not derivable by adding failsafe transformations of any Horn rules with two or more premises to REFLEX, RW, LCE, VCM.

PROOF. The positive part is proven in the text. For the negative part, we define a probabilistic property that holds for REFLEX, RW, LCE, VCM and all failsafe transformations of Horn rules with two or more premises, is preserved under chaining, but fails for the rule WOR.

Let  $p$  be any probability function. We write  $p^*(x, a)$  for the two-argument function defined by putting  $p^*(x, a) = p_a(x)$  when  $p(a) \neq 0$  and  $p^*(x, a) = 0$  when  $p(a) = 0$ . Evidently,  $p^*$  is not a standard probability function, although it is closely related to conditionalization. We consider the following *majoration property* for Horn rules:

$$\text{if } p^*(x_i, a_i) \neq 0 \text{ for all } i \leq n, \text{ then } p^*(b, y) \geq p^*(x_i, a_i) \text{ for all } i \leq n$$

where  $b \vdash y$  is the conclusion of the rule and  $a_i \vdash x_i$  are its premises. Note the presence of two quantifiers in this condition, with disjoint scopes. For simplicity of notation, we are here assuming that the Horn rule has finitely many premises; the same argument goes through whatever the cardinality.

*Lemma 3.2.1.* The rule WOR fails the majoration property for some probability function  $p$ .

*Verification.* The premises of WOR are  $a \wedge b \vdash x$ ,  $a \wedge \neg b \vdash x$  and its conclusion is  $a \vdash x$ . Choose  $a, b, x$  to be distinct elementary letters. It suffices to find a probability function  $p$  such that  $p^*(x, a \wedge b) \neq 0$ ,  $p^*(x, a \wedge \neg b) \neq 0$ , and (say)  $p^*(x, a) < p^*(x, a \wedge b)$ . Define a function  $p$  on state descriptions defined by putting  $p(a \wedge b \wedge x) = p(a \wedge \neg b \wedge x) = p(a \wedge \neg b \wedge \neg x) = 1/3$ , all other state descriptions getting value 0, and consider its unique extension to a probability function on all formulae. Then  $p(a \wedge b) = 1/3$  and  $p(a \wedge \neg b) = 2/3$  so  $p^*(x, a \wedge b) = p(a \wedge b \wedge x)/p(a \wedge b) = (1/3)/(1/3) = 1 > 0$  and

$p^*(x, a \wedge \neg b) = p(a \wedge \neg b \wedge x)/p(a \wedge \neg b) = (1/3)/(2/3) = 1/2 > 0$ . Also  $p^*(x, a) = p(a \wedge x)/p(a) = (2/3)/1 = 2/3 < 1 = p^*(x, a \wedge b)$ .

*Lemma 3.2.2.* Every Horn rule derivable from REFLEX, RW, LCE, VCM plus any set of failsafe transformations of Horn rules with two or more premises, has the majoration property for every probability function  $p$ .

*Verification.* Consider the set  $C$  consisting of REFLEX, RW, LCE, VCM plus any set of failsafe transformations of Horn rules. Clearly they are all Horn rules. Hence the rules derivable from them are precisely those that can be obtained by chaining. As the relation  $\geq$  between reals is transitive, it thus suffices to show that each rule in  $C$  has the majoration property.

The verifications for REFLEX, RW, LCE are trivial. For failsafe transformations of rules with two or more premises, the majoration property holds vacuously, because  $p^*(x_i, a_i) = 0$  for the transformed premise. It remains to check VCM.

The premise of VCM is  $a \vdash x \wedge y$  and the conclusion is  $a \wedge x \vdash y$ . Suppose  $p^*(x \wedge y, a) \neq 0$ ; we need to show that  $p^*(y, a \wedge x) \geq p^*(x \wedge y, a)$ . In the case that  $p(a \wedge x) = 0$  we have LHS = 0; if also  $p(a) = 0$  then also RHS = 0, while if  $p(a) \neq 0$  then RHS =  $p(a \wedge x \wedge y)/p(a) = 0$  again. In the case that  $p(a \wedge x) \neq 0$ , LHS =  $p(a \wedge x \wedge y)/p(a \wedge x) \geq p(a \wedge x \wedge y)/p(a) =$  RHS and we are done. ■

*Facts used in the proof of Observation 4.1*

*Fact 4.1.1.* (Hawthorne [8]). All rules in the family  $\mathbf{P}$  are derivable from  $\mathbf{OU}\{\mathbf{AND}\}$ .

PROOF. There is only one to check: OR. Suppose  $a \vdash x$  and  $b \vdash x$ . We need to show that  $a \vee b \vdash x$ . Applying LCE to the second supposition,  $(a \vee b) \wedge b \vdash x$ , so  $(a \vee b) \wedge b \vdash x \vee a$  (RW); also  $(a \vee b) \wedge \neg b \vdash x \vee a$  (SUP); thus  $a \vee b \vdash x \vee a$  (WOR). Similarly, applying LCE to the first supposition,  $(a \vee b) \wedge a \vdash x$ , so  $(a \vee b) \wedge a \vdash x \vee \neg a$  (RW); also  $(a \vee b) \wedge \neg a \vdash x \vee \neg a$  (SUP); thus  $a \vee b \vdash x \vee \neg a$  (WOR). Putting these together we have  $a \vee b \vdash (x \vee a) \wedge (x \vee \neg a)$  (AND), so  $a \vee b \vdash x$  (RW). ■

*Fact 4.1.2.* (Bochman [4]): All closure conditions in family  $\mathbf{P}$  are derivable from  $\mathbf{OU}\{\mathbf{OR}, \mathbf{CM}\}$ .

PROOF. By Fact 3.1.1 it suffices to derive AND from  $\mathbf{OU}\{\mathbf{OR}, \mathbf{CM}\}$ . A derivation is given by Bochman [4] Theorem 6.7.1 points (1) and (4). Here we give a rather different derivation via a curious variant of AND.

First, note that  $\mathbf{OU}\{\text{OR}, \text{CM}\}$  implies the rule: whenever  $a \vdash x$  and  $a \vdash y$ , then  $a \wedge (x \vee y) \vdash x \wedge y$ . For suppose  $a \vdash x$  and  $a \vdash y$ . By LCE it will suffice to show that  $(a \wedge x) \vee (a \wedge y) \vdash x \wedge y$ , so by OR it will suffice to show that  $(a \wedge x) \vdash x \wedge y$  and  $(a \wedge y) \vdash x \wedge y$ . We show the former; the latter is similar. On the one hand, applying CM to our two suppositions gives  $a \wedge x \vdash y$ . On the other hand, by SUP,  $(a \wedge x) \wedge \neg x \vdash x$ . Applying WAND to these gives  $(a \wedge x) \vdash y \wedge x$  so by RW,  $(a \wedge x) \vdash x \wedge y$  as desired. Now derive AND using the above rule. Suppose  $a \vdash x$  and  $a \vdash y$ . We want to show that  $a \vdash x \wedge y$ . By the above rule,  $a \wedge (x \vee y) \vdash x \wedge y$ , so by RW,  $a \wedge (x \vee y) \vdash x \leftrightarrow y$ . But also by SUP,  $a \wedge \neg(x \vee y) \vdash x \leftrightarrow y$ . Applying WOR to these gives  $a \vdash x \leftrightarrow y$ . Applying the above rule again to  $a \vdash x \leftrightarrow y$  and  $a \vdash x$  we have  $a \wedge (x \vee (x \leftrightarrow y)) \vdash x \wedge (x \leftrightarrow y)$ , so that by LCE and RW,  $a \wedge (y \rightarrow x) \vdash x \wedge y$ . Similarly, applying the above rule to  $a \vdash x \leftrightarrow y$  and  $a \vdash y$  we have  $a \wedge (y \vee (x \leftrightarrow y)) \vdash x \wedge (x \leftrightarrow y)$ , so that by LCE and RW,  $a \wedge (x \rightarrow y) \vdash x \wedge y$ . Applying OR to these gives  $[a \wedge (y \rightarrow x)] \vee [a \wedge (x \rightarrow y)] \vdash x \wedge y$  so that finally by LCE  $a \vdash x \wedge y$  as desired. ■

*Fact 4.1.3.* (Adams [2], Bochman [4]). All closure conditions in family  $\mathbf{P}$  are derivable from  $\mathbf{OU}\{\text{CM}, \text{CT}\}$ .

PROOF. By Fact 3.1.1, it suffices to derive AND from  $\mathbf{OU}\{\text{CM}, \text{CT}\}$ . A derivation of essentially this is given by Adams [2] (answer to exercise 2\*a of section 7.2) and another by Bochman [4], Theorem 6.7.1 point (1). Here we give a variant derivation. Suppose  $a \vdash x$  and  $a \vdash y$ . Then  $a \wedge x \vdash y$  (CM). Also,  $(a \wedge x) \wedge \neg x \vdash x$  by SUP, so  $a \wedge x \vdash y \wedge x$  (WAND), so  $a \wedge x \vdash x \wedge y$  (RW). Applying CT to this and the supposition  $a \vdash x$  finally gives  $a \vdash x \wedge y$  as desired. ■

*Fact 4.1.4.* (Bochman [4]). The closure condition OR is derivable from  $\mathbf{OU}\{\text{CT}\}$ .

PROOF. A derivation is given by Bochman [4] Theorem 6.7.1 point (2). Again we give a variant. Suppose  $a \vdash x$  and  $b \vdash x$ . From  $a \vdash x$  we have  $(a \vee b) \wedge a \vdash x$  (LCE), so  $(a \vee b) \wedge a \vdash (x \vee b)$  (RW), but also  $(a \vee b) \wedge \neg a \vdash (x \vee b)$  (SUP), so  $(a \vee b) \vdash (x \vee b)$  (WOR). From  $b \vdash x$  we have by LCE that  $((a \vee b) \wedge (x \vee b)) \wedge (\neg x \vee b) \vdash x$ , but also  $((a \vee b) \wedge (x \vee b)) \wedge \neg(\neg x \vee b) \vdash x$  (SUP), so  $(a \vee b) \wedge (x \vee b) \vdash x$  (WOR). Putting these together with CT we have  $(a \vee b) \vdash x$ . ■

*Fact 4.1.5.* (Bochman [4], Example 6.7.1). The closure condition AND is not derivable from  $\mathbf{OU}\{\text{CT}\}$ .

PROOF. Let  $\vdash$  be the consequence relation defined by putting  $a \vdash x$  iff either  $a$  is classically inconsistent, or  $a$  is consistent with  $x$ . Then it is straightforward to check that each rule in  $\mathbf{OU}\{\mathbf{CT}\}$  is satisfied, but AND is not. ■

*Observation 4.1.6.* AND is not derivable from  $\mathbf{OU}\{\mathbf{CM}\}$ .

PROOF. The model is defined and given in a diagram in the text; we need to show that it fails AND but satisfies all rules in  $\mathbf{OU}\{\mathbf{CM}\}$ . The failure of AND is immediate:  $123 \vdash 12$  and  $123 \vdash 13$  but  $123 \not\vdash 1$  (since  $1 = 12 \cap 13$ ). The success of reflexivity is immediate:  $a \leq a$  so  $a \vdash a$ . For the remaining rules in  $\mathbf{OU}\{\mathbf{CM}\}$  the verifications of success are as follows.

For RW: Suppose  $a \vdash x \leq y$ . We consider the three cases of the definition. Case 1:  $a \leq x$ . Then  $a \leq x \leq y$  so  $a \leq y$  so  $a \vdash y$ . Case 2:  $1 \leq a \wedge x$  and  $a \neq 123$ . Then  $1 \leq a \wedge x \leq a \wedge y$  and so also  $a \vdash y$ . Case 3:  $1 \leq a \wedge x$  and  $x \neq 1$ . Then  $1 \leq x$ , so since  $x \leq y$  and  $x \neq 1$  we have  $y \neq 1$ , so  $a \vdash y$ .

For LCE: Immediate from the fact that we are working in a Boolean algebra.

For WAND: Suppose  $a \vdash x$  and  $a \wedge \neg y \vdash y$ . We need to show  $a \vdash x \wedge y$ . We split into three cases.

Case 1:  $a \leq x$ ,  $a \leq y$ . Then  $a \leq x \wedge y$  so  $a \vdash x \wedge y$  as desired.

Case 2:  $a \leq y$  but  $a \not\leq x$ . Then  $1 \leq a \wedge x$  so  $1 \leq a \leq y$  and  $1 \leq x$ , so  $1 \leq a \wedge (x \wedge y)$ . So it suffices to show that if  $a = 123$  then  $y \neq 1$ , which is immediate given  $a \leq y$ .

Case 3:  $a \not\leq y$ . Then  $a \wedge \neg y \not\leq y$  so since  $a \wedge \neg y \vdash y$  we have  $1 \leq (a \wedge \neg y) \wedge y = \emptyset$ : impossible.

For WOR: Suppose  $a \wedge b \vdash x$  and  $a \wedge \neg b \vdash x$ . We need to show  $a \vdash x$ . We split into four cases.

Case 1:  $a \wedge b \leq x$  and  $a \wedge \neg b \leq x$ . Then  $a \leq x$  so  $a \vdash x$  as desired.

Case 2:  $a \wedge b \leq x$  but  $a \wedge \neg b \not\leq x$ . Then  $1 \leq (a \wedge \neg b) \wedge x \leq a \wedge x \leq x$ . Since  $1 \leq a \wedge x$ , to show  $a \vdash x$  it will suffice to show that if  $a = 123$  then  $x \neq 1$ . Suppose that  $a = 123$  but  $x = 1$ : we derive a contradiction. Then  $b = a \wedge b \leq x = 1 \leq a \wedge \neg b \wedge x \leq \neg b$ , so  $b \leq \neg b$  so that  $\neg b = 123$  and thus  $a \wedge \neg b = 123$  while  $x = 1$ , so  $a \wedge \neg b \not\vdash x$  contradicting an initial supposition.

Case 3:  $a \wedge b \not\leq x$  but  $a \wedge \neg b \leq x$ . Similar to Case 2.

Case 4:  $a \wedge b \not\leq x$  and  $a \wedge \neg b \not\leq x$ . Then  $1 \leq a \wedge \neg b \wedge x$  and also  $1 \leq a \wedge b \wedge x$ , so  $1 \leq \neg b$  and also  $1 \leq b$  so  $1 \leq \emptyset$ : impossible.

For CM: Suppose  $a \vdash x$  and  $a \vdash y$ . We need to show  $a \wedge x \vdash y$ . We split into three cases.

Case 1:  $a \leq x$ . Then  $a \wedge x = a \vdash y$  as desired.

Case 2:  $a \leq y$ . Then  $a \wedge x \leq y$  so  $a \wedge x \sim y$  as desired.

Case 3:  $a \not\leq x$  and  $a \not\leq y$ . Then  $1 \leq a \wedge x \leq x$  and  $1 \leq a \wedge y \leq y$  so  $1 \leq (a \wedge x) \wedge y$ . So to show  $a \wedge x \sim y$  it will suffice to show that if  $a \wedge x = 123$  then  $y \neq 1$ . But if  $a \wedge x = 123$  then  $a = 123$ , and since  $a \sim y$  we have  $y \neq 1$  as desired. ■

*Observation 7.2.* (Hawthorne [8], [9]). For  $n \geq 1$ , FS-PREF( $n+1$ ) and CL-PREF( $n+1$ ) (and also PREF( $n$ )) are probabilistically sound for all probability functions  $p$  and all thresholds  $t > n/n+1$ , although for each  $t \leq n/n+1$  there are  $p$  for which each rule fails.

PROOF. Since the rules CL-PREF( $n+1$ ) and PREF( $n$ ) are derivable from each other, we need only concern ourselves with FS-PREF( $n+1$ ) and CL-PREF( $n+1$ ). And since FS-PREF( $n+1$ ) implies CL-PREF( $n+1$ ), the soundness of FS-PREF( $n+1$ ) for  $t > n/n+1$  implies CL-PREF( $n+1$ ) is also sound these thresholds. Furthermore, the unsoundness of CL-PREF( $n+1$ ) at thresholds  $t \leq n/n+1$  implies the unsoundness of FS-PREF( $n+1$ ) for these thresholds. First we show that FS-PREF( $n+1$ ) is sound for all  $p$  and  $t > n/n+1$ . Then we'll show that each  $t \leq n/n+1$  there are  $p$  for which CL-PREF( $n+1$ ) fails.

Suppose that  $a \sim x_1, \dots, a \sim x_{n+1}, a \wedge (x_1 \wedge \dots \wedge x_{n+1}) \sim \perp$ , and  $a \not\sim \perp$ , where  $\sim_{p,t} = \sim$  is a probabilistic consequence relation for  $p$  at threshold  $t$ . We show that under these conditions  $t \leq n/n+1$ . Given the last supposition,  $p(a) > 0$  and  $t > 0$ , and so by the first  $n+1$  suppositions  $p_a(x_i) \geq t$  for each  $x_i$ . Hence, by the supposition  $a \wedge (x_1 \wedge \dots \wedge x_{n+1}) \sim \perp$  we have  $p(a \wedge (x_1 \wedge \dots \wedge x_{n+1})) = 0$  so  $p_a(x_1 \wedge \dots \wedge x_{n+1}) = 0$ . So  $1 = p_a(\neg(x_1 \wedge \dots \wedge x_{n+1})) \leq p_a(\neg x_1) + \dots + p_a(\neg x_{n+1}) = (1 - p_a(x_1)) + \dots + (1 - p_a(x_{n+1})) \leq (n+1) \cdot (1 - t)$ . So  $1 \leq (n+1) \cdot (1 - t)$ , i.e.,  $t \leq n/n+1$ .

To see that for any  $t \leq n/n+1$  there is a  $p$  such that CL-PREF( $n+1$ ) fails, just let  $t > 0$  have some fixed value  $\leq n/n+1$ , and observe that for elementary letters  $x_1, \dots, x_{n+1}$ , and letting ' $a$ ' be the formula ' $(\neg x_1 \vee \dots \vee \neg x_{n+1})$ ', there is clearly a  $p$  with the following properties:  $p(a) > 0$ , for all distinct  $x_i$  and  $x_j$ ,  $p_a(\neg x_i \wedge \neg x_j) = 0$ , and the  $p_a(\neg x_i)$  are all equal. For this  $t$  and  $p$  the rule fails, as follows. For each  $x_i$ ,  $p_a(\neg x_i) = 1/n+1$ , so  $p_a(x_i) = n/n+1 \geq t$ , so  $a \sim x_i$ . And  $a \vdash \neg(x_1 \wedge \dots \wedge x_{n+1})$ . But  $a \not\sim \perp$  since  $p(a) > 0$  and  $p(a \wedge \perp) = 0$ . ■

*Corollary 7.4.* Let  $t \in (1/2, 1)$ . Put  $n$  to be the largest positive integer such that  $t > n/n+1$  (i.e., the largest  $n$  such that  $n < t/1 - t$ ). Then FS-PREF( $n+1$ ) is strictly the strongest rule in the array of the Figure for Observation 7.3 that's sound for probabilistic consequence with threshold  $t$ .

PROOF. Consider the set of all rules in the array that are sound for probabilistic consequence with threshold  $t$ . From Observation 7.3, FS-PREF( $n+1$ ) implies all rules in this set. To show that it is strictly the strongest in the set, it suffices to show that it is not implied by CL-PREF( $n+1$ ). We show quite generally that FS-PREF( $m+1$ ) is not implied by CL-PREF( $m+1$ ), for any  $m \geq 1$ .

Let  $a$  be a tautology, and let  $x_1, \dots, x_{m+1}$  be elementary letters. It is not difficult to construct a probability function  $p$  with  $p(\neg x_i \wedge \neg x_j) = 0$  for each distinct pair  $x_i$  and  $x_j$ , and such that  $p(x_i) = m/m+1$  for each  $x_i$ . Put the threshold  $s = m/m+1$ . It will suffice to show that CL-PREF( $m+1$ ) holds while FS-PREF ( $m+1$ ) fails for the consequence relation  $\vdash$  determined by  $p$ ,  $s$  under Definition 2.2.

CL-PREF( $m+1$ ) holds vacuously since  $a \not\vdash \neg(x_1 \wedge \dots \wedge x_{m+1})$ . We show that FS-PREF( $m+1$ ) fails. On the one hand,  $p_a(x_i) = p(x_i) = m/m+1 = s$ , so each  $a \vdash x_i$  holds. Also  $p(a \wedge (x_1 \wedge \dots \wedge x_{m+1})) = p(x_1 \wedge \dots \wedge x_{m+1}) = 1 - p(\neg x_1 \vee \dots \vee \neg x_{m+1}) = 1 - (p(\neg x_1) + \dots + p(\neg x_{m+1})) = 1 - ((m+1) \cdot (1 - m/m+1)) = 0$ , so  $a \wedge (x_1 \wedge \dots \wedge x_{m+1}) \vdash \neg(x_1 \wedge \dots \wedge x_{m+1})$  holds. On the other hand,  $p(a \wedge \perp)/p(a) = 0 < s$ . ■

*Observation 7.5.* (Hawthorne [8]). For  $n \geq 1$ , FS-PLAUS( $n+1$ ) is probabilistically sound for all probability functions  $p$  and all thresholds  $t > 1/n+1$ , although for each  $t \leq 1/n+1$  there are  $p$  for which it fails.

PROOF. Suppose that  $a \vdash x_1, \dots, a \vdash x_{n+1}$  and  $a \not\approx \neg(x_1 \wedge x_2)$ ,  $a \not\approx \neg(x_1 \wedge x_3)$ ,  $\dots$ ,  $a \not\approx \neg(x_n \wedge x_{n+1})$ , and  $a \not\vdash \perp$ , where  $\vdash$  is a probabilistic consequence relation for  $p$  at threshold  $t$ . We show that under these conditions  $t \leq 1/n+1$ . Given the suppositions,  $p(a) > 0$  and  $t > 0$  and also  $p_a(x_i) \geq t$  for each  $x_i$ . Hence, since  $a \not\approx \neg(x_i \wedge x_j)$  for each distinct  $i$  and  $j$ , it follows that  $p(a \wedge (x_i \wedge x_j)) = 0$  for all distinct  $i$  and  $j$ ; so  $p_a(x_i \wedge x_j) = 0$ . Then  $1 \geq p_a(x_1 \vee \dots \vee x_{n+1}) = p_a(x_1) + \dots + p_a(x_{n+1}) \geq (n+1) \cdot t$ . So  $t \leq 1/n+1$ .

To see that for any  $t \leq 1/n+1$  there is a  $p$  such that FS-PLAUS( $n+1$ ) fails, just let  $t$  have some fixed value  $\leq 1/n+1$ , and observe that for elementary letters  $a, x_1, \dots, x_{n+1}$ , there is clearly a  $p$  with the following properties:  $p(a) > 0$ , and for each distinct  $x_i$  and  $x_j$ ,  $p_a(x_i \wedge x_j) = 0$ , and  $p_a(x_i) = p_a(x_j)$ . For this  $t$  and  $p$  the rule fails, as follows. For each  $x_i$ ,  $p_a(x_i) = 1/n+1 \geq t$ , so  $a \vdash x_i$ . For each distinct  $i$  and  $j$ ,  $p_a((x_i \wedge x_j)) = 0$ , so  $p(a \wedge \neg\neg(x_i \wedge x_j)) = 0$ , so  $a \wedge \neg\neg(x_i \wedge x_j) \vdash \neg(x_i \wedge x_j)$  (i.e.  $a \not\approx \neg(x_i \wedge x_j)$ ). But  $a \not\vdash \perp$  because  $p(a) > 0$  and  $p(a \wedge \perp) = 0$ . ■

*Observation 8.1.* CA-PREF( $t$ ) is probabilistically sound for all thresholds  $t$ .

PROOF. Let  $p$  be any probability function and let  $t$  be any threshold level such that (for given  $n$  and  $k_i \geq 1$ ,  $1 \leq i \leq n+1$ ) the corresponding probabilistic consequence relation  $\sim$  satisfies the following:

$a \sim x_{i,1}$ ,  $a \wedge x_{i,1} \sim x_{i,2}$ , ...,  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \sim x_{i,k_i}$ , for all  $1 \leq i \leq n+1$ , and  $\bigwedge_{i=1}^{n+1} (a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \sim \perp$ , but  $a \not\sim \perp$ . We show that  $n \geq \sum_{i=1}^{n+1} t^{k_i}$  follows.

From  $a \not\sim \perp$  we have  $t > 0$  and  $p(a) > 0$ . From  $a \sim x_{i,1}$  we have  $p(a \wedge x_{i,1})/p(a) \geq t > 0$  and  $p(a \wedge x_{i,1}) > 0$ . ... From  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \sim x_{i,k_i}$  we have  $p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1}) \geq t > 0$  and  $p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) > 0$ .

Now from  $\bigwedge_{i=1}^{n+1} (a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \sim \perp$  we have  $p(\bigwedge_{i=1}^{n+1} (a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})) = 0$ , so  $p_a(\bigwedge_{i=1}^{n+1} (x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})) = 0$ .

Then  $1 = p_a(\bigvee_{i=1}^{n+1} \neg(x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})) \leq \sum_{i=1}^{n+1} p_a(\neg(x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})) = (n+1) - \sum_{i=1}^{n+1} p_a(x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})$ .

So  $n \geq \sum_{i=1}^{n+1} p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a) = \sum_{i=1}^{n+1} [p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1})] \cdot [p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-2})] \cdot \dots \cdot [p(a \wedge x_{i,1})/p(a)] \geq \sum_{i=1}^{n+1} t^{k_i}$ .

Thus,  $n \geq \sum_{i=1}^{n+1} t^{k_i}$ .  $\blacksquare$

*Observation 8.2.* CA-PLAUS( $t$ ) is probabilistically sound for all thresholds  $t$ .

PROOF. Let  $p$  be any probability function and let  $t$  be any threshold level. Suppose that (for given  $n$  and  $k_i \geq 1$ ,  $1 \leq i \leq n+1$ ) the corresponding probabilistic consequence relation  $\sim$  satisfies the following:

$a \sim x_{i,1}$ ,  $a \wedge x_{i,1} \sim x_{i,2}$ , ...,  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \sim x_{i,k_i}$ , for all  $1 \leq i \leq n+1$ , and  $(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \wedge (a \wedge x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,k_j-1} \wedge x_{j,k_j}) \sim \perp$  for each  $i$  and  $j$  such that  $1 \leq i < j \leq n+1$ , but  $a \not\sim \perp$ . We show that  $1 \geq \sum_{i=1}^{n+1} t^{k_i}$  follows.

From  $a \not\sim \perp$  we have  $t > 0$  and  $p(a) > 0$ . From  $a \sim x_{i,1}$  we have  $p(a \wedge x_{i,1})/p(a) \geq t > 0$  and  $p(a \wedge x_{i,1}) > 0$ . ... From  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \sim x_{i,k_i}$  we have  $p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1}) \geq t > 0$  and  $p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) > 0$ .

Now from  $(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \wedge (a \wedge x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,k_j-1} \wedge x_{j,k_j}) \sim \perp$  we have  $p((a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \wedge (a \wedge x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,k_j-1} \wedge x_{j,k_j})) = 0$  (for each distinct  $i, j$ ), so  $p_a((x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \wedge (x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,k_j-1} \wedge x_{j,k_j})) = 0$  (for each distinct  $i, j$ ).

Then  $1 \geq p_a(\bigvee_{i=1}^{n+1} (x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})) = \sum_{i=1}^{n+1} p_a(x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) = \sum_{i=1}^{n+1} p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a) =$



$$\sum_{i=1}^{n+1} [p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) / p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1})] \cdot [p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1}) / p(ax_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-2})] \cdot \dots \cdot [p(a \wedge x_{i,1}) / p(a)] \geq \sum_{i=1}^{n+1} t^{k_i}.$$

Thus,  $1 \geq \sum_{i=1}^{n+1} t^{k_i}$ . ■

*Observation 9.2.* (Hawthorne [8], [9]). For  $n \geq 1$ , FS-LOTT( $n+1$ ) is probabilistically sound for all probability functions  $p$  and all thresholds  $t \leq n/n+1$ , although for each  $t > n/n+1$  there are  $p$  for which it fails.

PROOF. Suppose that  $a \wedge (x_1 \wedge x_2) \sim \neg(x_1 \wedge x_2)$ ,  $a \wedge (x_1 \wedge x_3) \sim \neg(x_1 \wedge x_3)$ , ...,  $a \wedge (x_n \wedge x_{n+1}) \sim \neg(x_n \wedge x_{n+1})$ , but  $a \not\sim \neg x_1$  and ... and  $a \not\sim \neg x_{n+1}$ , where  $\sim$  is a probabilistic consequence relation for  $p$  at threshold  $t$ . (We show that under these conditions  $t > n/n+1$ .) Then  $p(a) > 0$  (else  $a \sim \neg x_1$ ); for each  $x_i$ ,  $p_a(\neg x_i) < t$ ; and for distinct pair  $x_i$  and  $x_j$ ,  $p_a(x_i \wedge x_j) = 0$  [because  $p(\neg(x_i \wedge x_j) \wedge a \wedge (x_i \wedge x_j)) = 0$ , and  $a \wedge (x_i \wedge x_j) \sim \neg(x_i \wedge x_j)$ , so  $p(a \wedge (x_i \wedge x_j)) = 0$ , so  $p_a(x_i \wedge x_j) = 0$ ]. Then  $1 \geq p_a(x_1 \vee \dots \vee x_{n+1}) = p_a(x_1) + \dots + p_a(x_{n+1}) = (1 - p_a(\neg x_1)) + \dots + (1 - p_a(\neg x_{n+1})) > (n+1) \cdot (1-t)$ . So  $1 > (n+1) \cdot (1-t)$  - i.e.,  $t > n/n+1$ .

To see that for any  $t > n/n+1$  there is a  $p$  such that FS-PREF( $n+1$ ) fails, just let  $t$  have some fixed value  $> n/n+1$ , and observe that for elementary letters  $a, x_1, \dots, x_{n+1}$ , there is clearly a  $p$  with the following properties:  $p(a) > 0$ , and for each distinct  $x_i$  and  $x_j$ ,  $p_a(x_i \wedge x_j) = 0$ , and  $p_a(x_i) = p_a(x_j)$ . For this  $t$  and  $p$  the rule fails, as follows. For each  $x_i$ ,  $p_a(x_i) = 1/n+1$ , so  $p_a(\neg x_i) = n/n+1 < t$ , so  $a \not\sim \neg x_i$ . And for each distinct  $x_i$  and  $x_j$ , because  $p_a(x_i \wedge x_j) = 0$ , we have  $p(a \wedge (x_i \wedge x_j)) = 0$ , so  $a \wedge (x_i \wedge x_j) \sim \neg(x_i \wedge x_j)$ . ■

*Observation 9.3.* (Hawthorne [8]). For  $n \geq 1$ , FS-POSS( $n+1$ ) is probabilistically sound for all probability functions  $p$  and all thresholds  $t \leq 1/n+1$ , although for each  $t > 1/n+1$  there are  $p$  for which it fails.

PROOF. Suppose for distinct formulae,  $x_1, \dots, x_{n+1}$ ,  $a \wedge (x_1 \wedge \dots \wedge x_{n+1}) \sim \perp$ , but  $a \not\sim \neg x_1$  and ... and  $a \not\sim \neg x_{n+1}$ , where  $\sim$  is a probabilistic consequence relation for  $p$  at threshold  $t > 0$ . (We show that under these conditions  $t > 1/n+1$ .) Then  $p(a) > 0$  (else  $a \sim \neg x_1$ ); for each  $x_i$ ,  $p_a(\neg x_i) < t$  (and so  $t > 0$ ); and  $p_a(x_1 \wedge \dots \wedge x_{n+1}) = 0$  [because  $p(a \wedge (x_1 \wedge \dots \wedge x_{n+1}) \wedge \perp) = 0$ , so from  $a \wedge (x_1 \wedge \dots \wedge x_{n+1}) \sim \perp$  we have that  $p(a \wedge (x_1 \wedge \dots \wedge x_{n+1})) = 0$ ]. Then  $1 = p_a(\neg x_1 \vee \dots \vee \neg x_{n+1}) \leq p_a(\neg x_1) + \dots + p_a(\neg x_{n+1}) < (n+1) \cdot t$ . So  $1/(n+1) < t$ .

To see that for any  $t > 1/n+1$  there is a  $p$  such that FS-POSS( $n+1$ ) fails, just let  $t$  have some fixed value  $> 1/n+1$ , and observe that for elementary letters  $a, y_1, \dots, y_{n+1}$ , there is clearly a  $p$  with the following properties:

$p(a) > 0$ , and for each distinct  $y_i$  and  $y_j$ ,  $p_a(y_i \wedge y_j) = 0$ , and  $p_a(y_i) = p_a(y_j)$ . For this  $t$  and  $p$  the rule fails, as follows. For each  $y_i$ ,  $p_a(y_i) = 1/n+1 < t$ , so  $a \not\vdash y_i$  (i.e.  $a \not\vdash \neg y_i$ ). And  $p_a(\neg(\neg y_1 \wedge \dots \wedge \neg y_{n+1})) = p_a(y_1) + \dots + p_a(y_{n+1}) = 1 \geq t$ , so  $p(a \wedge \neg y_1 \wedge \dots \wedge \neg y_{n+1}) = 0$ , so  $a \wedge (\neg y_1 \wedge \dots \wedge \neg y_{n+1}) \vdash \perp$ . Now just take each ' $x_i$ ' in the rule to be ' $\neg y_i$ ', and we have our counterexample to the rule. ■

*Observation 9.6.* CA-LOTT( $t$ ) is probabilistically sound for all thresholds  $t$ .

PROOF. Let  $p$  be any probability function and let  $t$  be any threshold level such that (for given  $n$  and  $k_i \geq 1$ ,  $1 \leq i \leq n+1$ ) the corresponding probabilistic consequence relation  $\vdash$  satisfies the following:

$a \not\vdash \neg x_{i,1}$ ,  $a \wedge x_{i,1} \not\vdash \neg x_{i,2}$ , ...,  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \not\vdash \neg x_{i,k_i}$ , for all  $1 \leq i \leq n+1$ , and  $(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \wedge (a \wedge x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,k_j-1} \wedge x_{j,k_j}) \vdash \perp$  for each  $i$  and  $j$  such that  $1 \leq i < j \leq n+1$ . We show that  $1 > \sum_{i=1}^{n+1} (1-t)^{k_i}$  follows.

From  $a \not\vdash \neg x_{i,1}$  we have  $t > 0$ ,  $p(a) > 0$ , and  $p(a \wedge \neg x_{i,1})/p(a) < t$ , so  $p(a \wedge x_{i,1})/p(a) > 1-t$ . From  $a \wedge x_{i,1} \not\vdash x_{i,2}$  we have  $p(a \wedge x_{i,1}) > 0$  and  $p(a \wedge x_{i,1} \wedge \neg x_{i,2})/p(a \wedge x_{i,1}) < t$ , so  $p(a \wedge x_{i,1} \wedge x_{i,2})/p(a \wedge x_{i,1}) > 1-t$ . ... From  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \vdash x_{i,k_i}$  we have  $p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1}) > 0$  and  $p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge \neg x_{i,k_i})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1}) < t$ , so  $p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1}) > 1-t$ .

Now from  $(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \wedge (a \wedge x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,k_j-1} \wedge x_{j,k_j}) \vdash \perp$  we have

$p((a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \wedge (a \wedge x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,k_j-1} \wedge x_{j,k_j})) = 0$   
(for each distinct  $i, j$ ),

so  $p_a((x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \wedge (x_{j,1} \wedge x_{j,2} \wedge \dots \wedge x_{j,k_j-1} \wedge x_{j,k_j})) = 0$  (for each distinct  $i, j$ ).

Then  $1 \geq p_a(\bigvee_{i=1}^{n+1} (x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})) =$   
 $\sum_{i=1}^{n+1} p_a(x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) =$   
 $\sum_{i=1}^{n+1} p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a) =$   
 $\sum_{i=1}^{n+1} [p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1})]$   
 $\cdot [p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-2})] \cdot \dots \cdot [p(a \wedge x_{i,1})/p(a)]$   
 $> \sum_{i=1}^{n+1} (1-t)^{k_i}$ .

Thus,  $1 > \sum_{i=1}^{n+1} (1-t)^{k_i}$ . ■

*Observation 9.7.* CA-POSS( $t$ ) is probabilistically sound for all thresholds  $t$ .

PROOF. Let  $p$  be any probability function and let  $t$  be any threshold level such that (for given  $n$  and  $k_i \geq 1$ ,  $1 \leq i \leq n+1$ ) the corresponding probabilistic consequence relation  $\vdash$  satisfies the following:

$a \not\vdash \neg x_{i,1}$ ,  $a \wedge x_{i,1} \not\vdash \neg x_{i,2}$ , ...,  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \not\vdash \neg x_{i,k_i}$ , for all  $1 \leq i \leq n+1$ , and  $\bigwedge_{i=1}^{n+1} (a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \vdash \perp$ . We show that  $n > \sum_{i=1}^{n+1} (1-t)^{k_i}$  follows.

From  $a \not\vdash \neg x_{i,1}$  we have  $t > 0$ ,  $p(a) > 0$ , and  $p(a \wedge \neg x_{i,1})/p(a) < t$ , so  $p(a \wedge x_{i,1})/p(a) > 1-t$ . From  $a \wedge x_{i,1} \not\vdash x_{i,2}$  we have  $p(a \wedge x_{i,1}) > 0$  and  $p(a \wedge x_{i,1} \wedge \neg x_{i,2})/p(a \wedge x_{i,1}) < t$ , so  $p(a \wedge x_{i,1} \wedge x_{i,2})/p(a \wedge x_{i,1}) > 1-t$ . ...

From  $a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \vdash x_{i,k_i}$  we have  $p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1}) > 0$  and  $p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge \neg x_{i,k_i})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1}) < t$ , so  $p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1}) > 1-t$ .

Now from  $\bigwedge_{i=1}^{n+1} (a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i}) \vdash \perp$  we have  $p(\bigwedge_{i=1}^{n+1} (a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})) = 0$ , so  $p_a(\bigwedge_{i=1}^{n+1} (x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})) = 0$ .

Then  $1 = p_a(\bigvee_{i=1}^{n+1} \neg(x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})) \leq \sum_{i=1}^{n+1} p_a(\neg(x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})) = (n+1) - \sum_{i=1}^{n+1} p_a(x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})$ .

So  $n \geq \sum_{i=1}^{n+1} p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a) = \sum_{i=1}^{n+1} [p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1} \wedge x_{i,k_i})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1})] \cdot [p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-1})/p(a \wedge x_{i,1} \wedge x_{i,2} \wedge \dots \wedge x_{i,k_i-2})] \cdot \dots \cdot [p(a \wedge x_{i,1})/p(a)] > \sum_{i=1}^{n+1} (1-t)^{k_i}$ .

Thus,  $n > \sum_{i=1}^{n+1} (1-t)^{k_i}$ . ■

**Acknowledgements.** Parts of the research for this paper were carried out while the second author was a guest of National ICT Australia (NICTA) KRR Unit in Sydney, July-September 2005; MIMS, School of Mathematics, the University of Manchester, February-March 2006; the Laboratoire d'Informatique de Marseille, Université de Provence, April 2006; and the Chinese Academy of Sciences in Guiyang, China in May 2006. Thanks for their hospitality are due to Professors Toby Walsh and Norman Foo at NICTA, Jeff Paris at MIMS, Karl Schlechta in Marseille, and Zhang Mingyi in Guiyang. Also many thanks to Alexander Bochman, Michael Freund and Hannes Leitgeb for valuable comments on drafts.

## References

- [1] ADAMS, E. W. 'Four probability-preserving properties of inferences', *Journal of Philosophical Logic* 25: 1-24, 1996.
- [2] ADAMS, E. W. *A Primer of Probability Logic*, CSLI Publications, 1998.
- [3] BEZZAZI, H., D. MAKINSON, and R. P. PEREZ 'Beyond rational monotony: some strong non-Horn rules for nonmonotonic inference relations', *Journal of Logic and Computation* 7: 605-631, 1997.

- [4] BOCHMAN, A. *A Logical Theory of Nonmonotonic Inference and Belief Change*, Springer, 2001.
- [5] BURGESS, J. ‘Quick completeness proofs for some logics of conditionals’, *Notre Dame Journal of Formal Logic* 22: 76-84, 1981.
- [6] DUBOIS, D. and Henri Prade ‘Possibility theory, probability theory and multiple-valued logics: a clarification’, *Annals of Mathematics and Artificial Intelligence* 32: 35-66, 2001.
- [7] FREUND, M. ‘Injective models and disjunctive relations’, *Journal of Logic and Computation* 3: 231-247, 1993.
- [8] HAWTHORNE, J. ‘On the logic of nonmonotonic conditionals and conditional probabilities’, *Journal of Philosophical Logic* 25: 185-218, 1996.
- [9] HAWTHORNE, J. ‘Nonmonotonic conditionals that behave like conditional probabilities above a threshold’, to appear in *Journal of Applied Logic* 5, 2007.
- [10] KRAUS, S., D. LEHMANN, and M. MAGIDOR ‘Nonmonotonic reasoning, preferential models and cumulative logics’, *Artificial Intelligence* 44: 167-207, 1990.
- [11] KYBURG, H. E. *Probability and the Logic of Rational Belief*, Wesleyan University Press, 1961.
- [12] KYBURG, H. E., Choh Man TENG, and G. WHEELER ‘Conditionals and consequences’, to appear in *Journal of Applied Logic* 5, 2007.
- [13] LEHMANN, D. and M. MAGIDOR ‘What does a conditional knowledge base entail?’, *Artificial Intelligence* 55: 1-60, 1992.
- [14] MAKINSON, D. ‘The paradox of the preface’, *Analysis* 25: 205-207, 1965.
- [15] MAKINSON, D. ‘General patterns in nonmonotonic reasoning’, in Gabbay, Hogger, and Robinson (eds.), *Handbook of Logic in Artificial Intelligence and Logic Programming* vol. 3, 35-110, Oxford University Press, 1994.
- [16] MAKINSON, D. *Bridges from Classical to Nonmonotonic Logic*, Series: *Texts in Computing*, vol 5, College Publications, 2005.
- [17] STALNAKER, R. ‘A theory of conditionals’, in N. Rescher ed., *Studies in Logical Theory* Blackwell, 1968.
- [18] VAN BENTHEM, J. ‘Foundations of conditional logic’ *Journal of Philosophical Logic* 13: 303-349, 1984.

JAMES HAWTHORNE  
 Department of Philosophy  
 University of Oklahoma  
 605 Dale Hall Tower  
 Norman, OK 73019, USA  
 hawthorne@ou.edu

DAVID MAKINSON  
 Department of Philosophy, Logic, and Scientific Method  
 London School of Economics  
 Houghton Street  
 London, WC2A 2AE, UK  
 david.makinson@googlemail.com