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ON THE LOGIC OF NONMONOTONIC CONDITIONALS AND CONDITIONAL PROBABILITIES: PREDICATE LOGIC [1]

ABSTRACT. In a previous paper I described a range of nonmonotonic conditionals that behave like conditional probability functions at various levels of probabilistic support. These conditionals were defined as semantic relations on an object language for sentential logic. In this paper I extend the most prominent family of these conditionals to a language for predicate logic. My approach to quantifiers is closely related to Hartry Field's probabilistic semantics. Along the way I will show how Field's semantics differs from a substitutional interpretation of quantifiers in crucial ways, and show that Field's approach is closely related to the usual objectual semantics. One of Field's quantifier rules, however, must be significantly modified to be adapted to nonmonotonic conditional semantics. And this modification suggests, in turn, an alternative quantifier rule for probabilistic semantics.

KEY WORDS: Conditionals, Nonmonotonic Logic, Popper Functions, Probability, Semantics

INTRODUCTION

In a previous paper in this journal (Hawthorne, 1996) I described a range of nonmonotonic conditionals that behave like conditional probability functions at various levels of probabilistic support. These conditionals were all defined on languages for sentential logic. In this paper I will extend the semantics of the most prominent family of these nonmonotonic conditionals (the family $\mathbf{ER}$) to a language for predicate logic. The present paper, however, is intended to be self-contained, and will not presuppose that the reader is familiar with the previous paper. But I will briefly summarize some results of the earlier paper that help to motivate the present project.

A conditional, $\rightarrow$, is said to be *monotonic* if whenever '$C \rightarrow B$' holds, '$(C\&D) \rightarrow B$' must also hold. For a monotonic conditional the addition of new information to the antecedent of the conditional cannot undermine the support of the consequent already tendered by the original antecedent. The material conditional and the logical consequence relation are familiar examples of monotonic conditionals. A conditional, $\rightarrow$, is called *nonmonotonic* if the addition of new information to the antecedent can undermine its (degree of) support for the consequent. Indicative and subjunctive conditionals are typically nonmonotonic. Conditional probability is also nonmonotonic in the sense that $P[B \mid D\&C]$ may have quite a different probability value than $P[B \mid C]$, either higher or lower.

In the previous paper I first characterized a range of distinct families of nonmonotonic conditionals on sentential languages in terms of semantic rules that the conditionals of each family obey. Then, I showed that the conditionals in each family are essentially just the *probability greater than or equal to r* parts of the Popper Functions. That is, I showed that for each specified family of conditionals there is a probability value r for that family such that, for each conditional $\rightarrow$ in the family there is a conditional probability function $P_{\rightarrow}$ (a Popper Function) that *agrees with* $\rightarrow$ in the following sense: for all sentences C and B, '$C \rightarrow B$' holds *just in case* $P_{\rightarrow}[B \mid C] \geqslant r$. Conversely, each family of conditionals may be completely generated by first choosing the appropriate level r for the family, and then specifying, for each conditional probability function (i.e. Popper Function) P, a conditional $\rightarrow_P$ generated from P as follows: for all pairs of sentences C and B, '$C \rightarrow_P B$' is defined to hold just when $P[B \mid C] \geqslant r$. Thus, each family of conditionals turns out to be just the collection of the *probability greater than or equal to r* parts of the Popper Functions (for an appropriate value of r for the family).

Of particular interest is the family of conditionals I called $\mathbf{ER}$ in the previous paper. The conditionals in $\mathbf{ER}$ are a particularly tenacious variety of nonmonotonic conditional, a variety in which the support of B by C may only be defeated by conjoining to C the kind of new information D that is "highly unexpected" in the sense that $\neg D$ is also supported by C (before D is added to C) – i.e. the following semantic rule holds for conditionals in $\mathbf{ER}$: if $C \rightarrow B$ and $C \nrightarrow \neg D$, then $C\&D \rightarrow B$. We saw in the previous paper that these relations turn out to be identical to the conditionals Lehmann and Magidor (1992) call the *Rational Consequence* relations. And, remarkably, the conditionals in $\mathbf{ER}$ turn out to be the conditionals that correspond to the probability 1 parts of the Popper Functions; they are essentially the same conditionals identified by McGee (1994).

In the previous paper I specified *autonomous semantic rules* for the family of conditionals $\mathbf{ER}$, and then I showed that $\mathbf{ER}$ contains just those conditionals $\rightarrow_P$ such that P is a Popper Function, and such that '$C \rightarrow_P B$' holds just when $P[B \mid C] = 1$. Of special significance is the fact that in the semantics for $\mathbf{ER}$ the nonmonotonic conditionals, $\rightarrow$, *are* the semantic primitives of the semantic theory, much as truth-value assignments are the semantic primitives in the more usual truth-value semantics for sentential languages. The conditionals, $\rightarrow$, belong to the metalanguage rather than the object language. The semantic rules for these conditionals do not presuppose other semantic notions like *truth* or *logical truth*. Rather, the notions of *logical truth* and *logical consequence* are independently definable in terms of the conditionals in **ER**, just as these logical notions are independently definable in terms of truth-value assignments.

It should not be too surprising that this is possible. Beginning with the work of Harper (1974, 1975), a number of logicians (e.g. Field (1977), Leblanc (1979), and van Fraassen (1981)) have investigated probabilistic semantic theories in which the Popper Functions, rather than truth-values, are the semantic primitives. This kind of semantic theory specifies the set of (possible) conditional probability functions (i.e. I call this set '**POPPERFN**') in terms of semantic rules that do not presuppose the notions of *truth* or *logical truth*, much as a truth-value semantics specifies the set of (possible) truth-value assignments (i.e. **TVA**) in terms of semantic rules. The probabilistic semantic theory provides an alternative semantic basis for the definition of notions of *logical truth* and *logical consequence*, which prove to agree (extensionally) with the usual (truth-value based) notions. In this respect the semantics of **ER** does with conditionals precisely what the probabilistic semantic theories have done with the Popper Functions.

Harper (1974, 1983), Field (1977), and Leblanc (1979, 1983a,b), also show how to specify probabilistic semantic theories for the language of predicate logic. However, each of these logicians extends the Popper Functions to quantifiers in a different way. Harper's and Leblanc's approaches are closely related to the truth-value semantics for substitutional quantification. Field's approach may look a bit like substitutional quantification, too; but on closer inspection we will see that Field's approach is quite closely tied to an objectual semantics for quantifiers.

The point of the present paper, then, is to pursue two related goals. The first goal is to extend the semantics of the nonmonotonic conditionals in the family **ER** to the language of predicate logic. My approach will be similar to the way that Field extends the probabilistic semantics of the Popper Functions to quantifiers. Along the way I will attempt to show how Field's approach differs from a semantics for substitutional quantifiers in crucial ways. It will turn out, however, that one of Field's probabilistic semantic rules for quantifiers is not precisely adaptable to the semantics of nonmonotonic conditionals. Rather, the way in which we must adjust Field's approach to apply it to nonmonotonic conditionals will suggest, in turn, an alternative to one of Field's quantifiers rules for probabilistic semantics. The second goal of the present paper, then, will be to take a cue from the conditional semantics, and investigate an alternative probabilistic semantic rule for quantifiers. I will carefully compare this alternative semantic quantifier rule to Field's rule.

Here is an outline of what follows. In Section 1 I extend the semantics of truth-value assignments, **TVA**, to a truth-value semantics for predicate logic. The semantic quantifier rules for the truth-value semantics will not employ the usual objectual apparatus of domains of discourse, but will still manage to specify the same set of possible truth-value assignments to sentences as objectual semantics. The quantifier rules for this truth-value semantics will serve as models for the quantifier rules in the semantic theories for conditionals and conditional probabilities developed in subsequent sections.

In Section 2 I will supplement the semantic rules for the nonmonotonic conditionals in **ER** with semantic rules for quantifiers that are closely analogous to the quantifier rules for the truth-value semantics. I will define a notion of logical consequence in terms of conditionals (rather than truth-values), and show that this notion is extensionally equivalent to the standard notion of logical consequence for predicate logic.

In Section 3 I will extend the probabilistic semantics for sentential languages, the semantics of **POPPERFN**, to a probabilistic semantics for predicate logic. The approach I will adopt employs a quantifier rule closely related to the quantifier rule for conditionals, a rule that looks distinctly weaker than the rule employed by Field (1977). This semantics will also furnish a basis for a notion of logical consequence that is extensionally equivalent to the standard notion.

In Section 4 I will compare the probabilistic semantic quantifier rules of Section 3 to Field's quantifier rule. Section 5 concludes the paper with some suggestions about ways in which conditional semantics and probabilistic semantics may be further extended.

One advantage of the general approach to quantifiers developed by Field over a more straightforwardly objectual semantics is that Field's approach manages to neatly side-step some of the metaphysical issues that would arise if one attempted to specify semantic theories for nonmonotonic conditionals and conditional probabilities in an objectual way. A quite natural reading of a nonmonotonic conditional assertion '$C \rightarrow B$' is, "in almost all worlds in which C is true, B is also true." And a natural reading of a conditional probabilistic assertion '$P[B \mid C] = r$' is, "the measure among worlds in which C is true of the class of worlds in which B is true is r." So, for example, when sentences C and B contain a name in common, a fully objectual version of nonmonotonic conditional semantics and of probabilistic semantics will presuppose some view on the cross-world identification of objects and the naming of objects across worlds. Field's approach to quantifiers neatly side-steps such issues. Although I believe that the metaphysical treatment required to provide more directly objectual accounts is worth pursuing, Field's approach permits us to get on with the task of exploring the logics of nonmonotonic conditionals and conditional probabilities without having to take a stand on these metaphysical issues in advance.

1. TRUTH-VALUES

Let $\mathcal{L}$ be a standard first-order language with a countable list of predicate symbols (including n-ary relation symbols), a countable list of name symbols, and a countably infinite list of individual variables. The primitive logical symbols of the language are the universal quantifiers (e.g. '(x)' for variable 'x'), and the conjunction and negation symbols '&' and '$\neg$'. Existential quantifiers and the other logical connectives are defined in the standard way in terms of these. (In Section 5 I will add '$=$' for identity as a logical symbol.) Formulas and sentences of $\mathcal{L}$ are as usual. Expressions such as the following will be used as metalinguistic terms for formulas of $\mathcal{L}$: $A, Bc, Fx, (x)Dxc_i, \neg C, (A\&B)$. On occasion I will need to refer to long conjunctions. I will abbreviate conjunctions of form '$(\cdots(B_1\&B_2)\&\cdots\&B_n)$' by suppressing parentheses, and write '$(B_1\&\cdots\&B_n)$' instead.

To describe the semantics for quantifiers I will eventually need to draw on a family of first-order languages that are name-extensions of the root language $\mathcal{L}$. I will call these languages *$\mathcal{L}$-languages*. Given any specific language $\mathcal{L}$, a language L is called an *$\mathcal{L}$-language* just in case L results from supplementing $\mathcal{L}$ with a countable number (possibly 0) of new names. All $\mathcal{L}$-languages have precisely the same syntax as described for $\mathcal{L}$, and differ from one another only with respect to the name symbols they contain. For each $\mathcal{L}$-language L, S_L is the set of closed sentences of L.

The usual *objectual* way to extend the semantics of sentential logic to quantifiers is to define the notion of *truth* in terms of a more primitive semantic predicate, *satisfaction*. Each interpretation of the language assigns an object from a domain of objects to each name symbol, and assigns a set of n-tuples of objects from the domain to each n-ary predicate symbol (the n-tuples of objects that *satisfy* the predicate). Satisfaction for complex formulas (under an interpretation) is defined in a recursive way that depends only on which objects from the domain satisfy the n-ary predicates and which objects the names name. A formula with n distinct free variables is defined as *true under an interpretation* just when it is satisfied by all n-tuples of objects (assigned to free variables) from the domain of the interpretation. Thus, in the usual objectual semantics for predicate logic the truth-value of a closed sentence under an interpretation ultimately depends on which n-tuples of objects are members of the sets of n-tuples associated with the n-ary predicates, and on which objects are assigned to the names.

Rather than appealing to domains of objects (and to assignments of objects and n-tuples of objects to names and predicates, respectively), there are other ways to provide for the assignment of truth-values to first-order sentences. One way to directly extend the semantics of Truth-Value Assignments, $\mathbf{TVA}$, for a sentential language is through a *substitutional interpretation* of the quantifiers, as follows (where '$v[B]$' abbreviates '$B \in v$')[2]:

For any $\mathcal{L}$-language $L, v \in \mathbf{SUB}_L$ (the set of $\mathbf{SUB}$stitutional quantifier truth-value assignments on L) *iff* $v \subseteq S_L$ such that:

1) v satisfies the rules of $\mathbf{TVA}$ applied to sentences of L (i.e. for all B and C in S_L: $v[\neg B]$ *iff* not $v[B]$; $v[(B \& C)]$ *iff* $v[B]$ and $v[C]$);
   for any formula Fx containing only x free:
2) *if* $v[(x)Fx]$, *then* $v[Fc]$ for all names c in L;
3) *if* $v[Fc]$ for all names c in L, *then* $v[(x)Fx]$.
   B is SUB_L*-logically true iff* for all $v \in \mathbf{SUB}_L,\ v[B]$.
   B is a SUB_L*-logical consequence* of a set of sentences Γ *iff* for all $v \in \mathbf{SUB}_L$, if $v[C]$ for each $C \in \Gamma$, then $v[B]$.

The semantics of $\mathbf{SUB}_L$ is not the $\mathbf{T}$ruth-value semantics for $\mathbf{Q}$uantifiers, $\mathbf{TQ}$, that I will ultimately adopt, but it points in the right direction. Whereas $\mathbf{SUB}_L$ interprets quantifiers substitutionally, the semantics for $\mathbf{TQ}$ will be more closely aligned with an objectual interpretation of quantifiers. However it will be easier to understand the rationale behind the semantics for $\mathbf{TQ}$ by first considering how $\mathbf{SUB}_L$ fails to adequately capture certain intuitions that underlie objectual semantics. This is not to say that there is anything intrinsically wrong with substitutional quantification. For some purposes we may want quantifiers to be read substitutionally. But the objectual reading of quantifiers is clearly more natural in many contexts, especially in mathematics and the sciences. (See Leblanc (1983b) for an excellent, comprehensive treatment of substitutional quantification for both truth-value semantics and probabilistic semantics.)

The $\mathbf{SUB}_L$ semantics fails to *model* the objectual notion of truth because of the inadequacy of rule 3. This rule is inadequate as a representation of the intuitive idea that universal quantifiers range over *all* objects under discussion, including any unnamed objects. Rule 3 only says that $(x)Fx$ is true in v if F is true of all named things; so $(x)Fx$ may be true in v even when some unnamed object fails to satisfy F. Indeed, in a substitutional semantics there is no way to say *anything* about unnamed objects. One might attempt to overcome this difficulty by adding enough names to the language to name every object. But when more than a countable number of things are under discussion (e.g. the geometric points in physical space) not enough new names can be added unless the language is uncountable, in which case its sentences are not recursively specifiable.

The logic of substitutional quantification expressed by $\mathbf{SUB}_L$ does *emulate* the logic of our usual understanding of quantifiers to a certain extent. It does turn out to specify the same set of *logical truths* as objectual semantics. But $\mathbf{SUB}_L$ fails to *fully emulate* the logic of objectual quantifiers; the infinite set of sentences $\{Fc_i \mid \text{ for every name } c_i \text{ in } L\}$ SUB$_L$-logically entails $(x)Fx$, but no subset of it does. Thus, compactness fails for the notion of logical consequence that accompanies substitutional quantifiers. This is symptomatic of the failure of the semantics of substitutional quantification to adequately reflect the objectual intuitions that underlie the standard notion of logical consequence.

Dunn and Belnap (1968) suggest a way to modify the $\mathbf{SUB}_L$ semantics to get the same logical consequence relation as objectual semantics, and yet maintain the substitutional interpretation of quantification. They accomplished this by defining logical consequence in terms of a language L together with *name-extensions* of L, extensions of L to languages with additional names.

DEFINITION 1. Let L be any first-order language. A *name-extensions* of L is any language that is just like L except that it may contain a countable number (perhaps 0) of new name symbols.

Given the way I specified the notion of an $\mathcal{L}$-language above, all $\mathcal{L}$-languages are name-extensions of $\mathcal{L}$; and for each $\mathcal{L}$-language L, all name-extensions of L are also $\mathcal{L}$-languages. For each language L^+ that is a name-extension of an $\mathcal{L}$-language L, $\mathbf{SUB}_{L^+}$ can be defined on L^+ in just the way that $\mathbf{SUB}_L$ was defined on L. The semantic rule for quantifiers on the $\mathcal{L}$-languages remains strictly substitutional. However, Dunn and Belnap suggest a modification of the definition of logical consequence (and, similarly, of logical truth) for each $\mathcal{L}$-language L, as follows:

B is a *SUB-logical consequence* of a set of sentences Γ *iff* for all L^+ a name-extension of L and all $v \in \mathbf{SUB}_{L^+}$, if $v[C]$ for each $C \in \Gamma$, then $v[B]$.

This definition produces the same *logical consequence relation* as the usual objectual semantics. And so compactness no longer fails – e.g., the infinite set of sentences $\{Fc_i \mid \text{ for every name } c_i \text{ in } L\}$ does not *SUB-logically entail* $(x)Fx$; for, there is a name-extension L^+ of L with new name b such that Fb is not in $\{Fc_i \mid \text{ for every name } c_i \text{ in } L\}$.

Although the invocation of extended languages in the $\mathbf{SUB}$ definition of logical consequence yields the same logical consequence relation (extensionally) as objectual semantics, it does not yield the same collection of possible truth-value assignments. One consequence of this is that *logical consistency* is not quite the same on Dunn and Belnap's substitutional account as it is on the objectual account. A set of sentences Γ is objectually logically consistent just in case there is a possible truth-value assignment to *just the language of the sentences* in Γ that makes all sentences in Γ true. But for $\Gamma = \{\neg(x)Fx, Fc_1, Fc_2, \ldots\}$, no possible substitutional truth-value assignment to *just the language of the sentences* in Γ can make all sentence in Γ true. Γ can only be ruled logically consistent in substitutional semantics relative to some substitutional truth-value assignment on a name-extension of the syntax that occurs in Γ.

The substitutional approach to quantifiers engendered in the $\mathbf{SUB}_L$ definition of the truth-value assignments circumvents the usual apparatus of objectual semantics (e.g. domains of discourse, assignments of objects to variables, names, and predicates, and the notion of *satisfaction*), but I think it does so at too high a price. It fails to represent all of the objectual truth-value assignments, and so fails to adequately capture the attendant notion of logical consistency. There is, however, a way to characterize *all* of the objectual truth-value assignments on $\mathcal{L}$-languages without employing the usual objectual apparatus. The basic idea is this.

Consider an objectual interpretation on language L that makes $(\exists x)Fx$ true and makes Fc false for each name c. Let v be the set of sentences that are true under that interpretation. Consider a name-extension L^+ of L which supplements L with a new name b. The original objectual interpretation can be name-extended to language L^+ in such a way that the previously unnamed object that satisfies F gets the new name b. This extended interpretation gives rise to a truth-value assignment v^+ that extends v (i.e. v^+ agrees with v on the language L) but also makes Fb true. Similarly, if every possible name-extension of a given objectual interpretation to an extended language (with additional names) makes Fc

true for *every* name c (old and new), then the original objectual interpretation (and all of its name-extensions) must make $(x)Fx$ true. Thus, each truth-value assignment that arises from an objectual interpretation can be associated with a whole collection of possible truth-value assignments that arise from name-extensions of the objectual interpretation to the extended languages.

Name-extensions of truth-value assignments play a central role in what follows, so I will take the trouble of stating the relevant definition precisely. We define what it means for an arbitrary *set of sentences* to be a *name-extension* of another *set of sentences* when the members of the first set belong to a language that is a name-extension of the language of the second set:

DEFINITION 2. For any set v a subset of S_L of an $\mathcal{L}$-language L, if L^+ is a name-extension of L and v^+ is a subset of S_{L+}, then v^+ is a *name-extension* of v just in case $v^+ \cap S_L = v$ (v^+ contains just the L sentences that v contains).

In Definition 2 I do not presuppose that v and v^+ are truth-value assignments, for I intend to employ the notion of a name-extension in a definition of 'truth-value assignment'.

One way, then, that we might specify an objectual truth-value assignment without drawing on the apparatus of objectual semantics is to directly define the associated class of truth-value assignments that would arise from a standard objectual interpretation and its name-extensions. The following definition formalizes this idea.

DEFINITION 3. R is a *TQ-model class if and only if* for all $v \in R,\ v$ is a subset of the sentences of some $\mathcal{L}$-language L and satisfies the following conditions (where '$v[A]$' abbreviates '$A \in v$'):

1) v satisfies the rules of $\mathbf{TVA}$ applied to sentences of L (i.e. for all B and C in S_L: $v[\neg B]$ *iff* not $v[B]$; $v[(B\&C)]$ *iff* $v[B]$ and $v[C]$);

   for any formula Fx in L containing only x free:

2) *if* $v[(x)Fx]$ *then* $v[Fc]$ for all names c in L;
3) *if* for each $v^+ \in R$ such that v^+ is name-extension of v to L^+ (where L^+ is a name-extension of L), for all names c in L^+, $v^+[Fc]$, *then* $v[(x)Fx]$.

Each TQ-model class is a collection of assignments of truth-values to some collection of $\mathcal{L}$-languages. Most TQ-model classes R contain a hodgepodge of many unrelated truth-value assignments; but ignore the *messiness*, since it will play no meaningful role. What is important is that each v in R has one (itself) or more name-extensions in R. Notice, too, that any two distinct name-extensions of v in R *may* disagree on truth-values for sentences containing names not in the language of v. However, the crucial point is that each v in R belongs to a nested chain of (one or more) name-extensions of itself (all in R), a chain in which each successor agrees with all of its predecessors on their common language. Some TQ-model classes consist only of name-extensions of a single truth-value assignment v. Each of these classes *correspond* to a class of truth-value assignments that arises from name-extensions of a single objectual interpretation – and it is these classes the supply the crucial link to objectual semantics. However, Definition 3 also permits (indeed, requires) that the union of any number of TQ-model classes is also a TQ-model class – and thus, the *messiness* of most of the classes, R. Thus, most TQ-model classes may be partitioned into a number of unrelated TQ-model subclasses, where all truth-value assignments within a given subclass are name-extensions of a single truth-value assignments in that subclass.

Notice, rule 3 requires that if $v[\neg(x)Fx]$, then at least one name-extension v^{+} of v in R has a name c in its language such that $v^{+}[\neg Fc]$. Similarly, if $v^{+}[\neg(y)Gy]$ for another sentence $(y)Gy$ in L^{+}, then v^{+} in turn has a name-extension v^{++} in R such that $v^{++}[\neg Gb]$ for some name b in L^{++}. Thus, the sentences in any finite set of negated universally quantified sentences (or set existentially quantified sentences) that belong to a truth-value assignment v in a TQ-model class R will all be exemplified by names in some single name-extension of v that also belongs to R.

Let's examine the connection between TQ-model classes and objectual truth-value assignments even more closely. First we can see that every objectual truth-value assignment is a member of a TQ-model class. For, let u be any objectual truth-value assignment defined on a $\mathcal{L}$-language L in the usual way in terms of some objectual interpretation I. It is well known that each objectual interpretation I may be extended to an ω-complete interpretation I^{+} – i.e. an interpretation I^{+} just like I but on a language L^{+} that extends L with a countable collection of new names such that I^{+} makes an instance $\neg Fc$ true whenever I^{+} makes $\neg(x)Fx$ true. The truth-value assignment u^{+} determined by I^{+} agrees with u on the sentences of L, and for any formula Fx with only x free, $u^{+}[\neg(x)Fx]$ just in case $u^{+}[\neg Fc]$ for some name c in L^{+}. It's routine to verify that $\{u, u^{+}\}$ satisfies the rules for a TQ-model class. Thus, every objectual truth-value assignment u belongs to some TQ-model class.

Conversely, every member of a TQ-model class is a truth-value assignment for some objectual interpretation. To see this, let R be any TQ-model class, let v be in R, and let L be the language of v. *Suppose for the moment that* any such v has an ω-complete name-extension v^+ on a language L^+ in *some* TQ-model class – i.e. suppose that for any formula Fx with only x free, $v^+[\neg(x)Fx]$ just in case $v^+[\neg Fc]$ for some name c in L^+. (I will discharge this supposition by proving it in a moment. Note that v^+ is not necessarily a member of R.) Then we employ the standard Henkin construction, letting the domain of interpretation consist of the set of names in L^+; let each name refer to itself; let the extension of each n-ary predicate B be the set of n-tuples of names $\langle c_1, c_2, \ldots, c_n\rangle$ such that $v^+[Bc_1c_2\ldots c_n]$. It is easy to verify that this objectual interpretation reproduces the set v^+ as its truth-value assignment. And v is just v^+ restricted to the language L. Thus, v is also an objectual truth-value assignment.

The argument just given rests on the assumption that any member v of a TQ-model class has an ω-complete name-extension belonging to *some* TQ-model class. To see that the assumption holds, begin with any v on a language L for some TQ-model class R. Let $v_1, v_2, \ldots$, be a nested sequence of name-extensions of v in R (each a name-extension of the preceding ones) on languages $L_1, L_2, \ldots$ (each a name-extension of the preceding ones) such that for any formula Fx (with only x free) in L_i, if $v_i[\neg(x)Fx]$, then for some $j \geqslant i,\ v_j[\neg Fc]$ for name c in L_j. Rule 3 for R guarantees that such a sequence of name-extensions of v exists. Let the language L_∞ be the union of the languages L_i, and let v_∞ be the union of the sequence of v_i. Clearly v_∞ satisfies rules 1 and 2 for members of TQ-model classes (if it didn't, then v_∞ would have to violate one of these rules for some sentences in some L_i; but v_i and v_∞ agree). Then $\{v_\infty\}$ is a TQ-model class: if $v_\infty[\neg(x)Fx]$, then for some v_i, $v_i[\neg(x)Fx]$; so for some $j \geqslant i,\ v_j[\neg Fc]$ (c in L_j); thus $v_\infty[\neg Fc]$. And notice that v_∞ is an ω-complete name-extension of v (although v_∞ might not be in R itself).

Thus, we see that each TQ-model class R is a collection of objectual truth-value assignments, and each objectual truth-value assignment is in some TQ-model class. Now define $\mathbf{TQ}$ as the union of all TQ-model classes.

DEFINITION 4. $\mathbf{TQ}$ is the union of all TQ-model classes defined on $\mathcal{L}$-languages.

Clearly $\mathbf{TQ}$ is itself a TQ-model class, the largest such class. And the arguments just presented have established the following theorem.

THEOREM 1. $\mathbf{TQ}$ *is the set of all objectual truth-value assignments on* $\mathcal{L}$*-languages.*

We may now define TQ-logical truth and TQ-logical consequence in the obvious way.

DEFINITION 5. B is a *TQ-logical truth* (i.e. $\vDash B$) *iff* for all $\mathcal{L}$-languages L such that $B \in S_L$ and all $v \in \mathbf{TQ}$ defined on L, $v[B]$. B is a *TQ-logical consequence* of a set of sentences Γ (i.e. $\Gamma \vDash B$) *iff* for all $\mathcal{L}$-languages L such that $B \in S_L$ and $\Gamma \subseteq S_L$ and for all $v \in \mathbf{TQ}$ defined on L, if $v[C]$ for each $C \in \Gamma$, then $v[B]$.[3]

Theorem 1 makes it obvious that TQ-logical truth and TQ-logical consequence precisely coincide with their objectual counterparts.

The main appeal of the semantics of $\mathbf{TQ}$ derives from its close connection to objectual semantics. The semantics for $\mathbf{TQ}$ not only captures the same logical truths and the same logical consequence relation as objectual semantics. What makes it really interesting is that it emulates the treatment of quantifiers by objectual semantics without directly appealing to a domain of objects, but, rather, by capturing objectual intuitions about the *possibility of naming* additional objects. For my purposes in this paper the semantics of $\mathbf{TQ}$ is mainly of interest as a bridge between standard objectual semantics and the semantic theories of conditionals and conditional probabilities I will describe in subsequent sections. $\mathbf{TQ}$'s treatment of quantifiers is easily adapted to extend the sentential language semantic theories $\mathbf{ER}$ and $\mathbf{POPPERFN}$ to quantifiers. And $\mathbf{TQ}$'s tie to objectual intuitions strongly suggests that the very similar semantic theories I will employ in the quantified versions of $\mathbf{ER}$ and $\mathbf{POPPERFN}$ are not far removed from a more directly objectual treatment (e.g. one that employs possible worlds and possible objects).

2. NONMONOTONIC CONDITIONALS

The treatment of quantifiers in the extension of $\mathbf{TVA}$ to $\mathbf{TQ}$ is precisely analogous to the method I will use in this section to extend the semantics of $\mathbf{ER}$ to the nonmonotonic Entailment-Relation semantics for Quantifiers, $\mathbf{EQ}$. The object-languages for $\mathbf{EQ}$ belongs to a family of languages for predicate logic consisting of a language $\mathcal{L}$ and its name-extensions, the $\mathcal{L}$-languages described in the previous section. A nonmonotonic conditional, $\rightarrow$, on an $\mathcal{L}$-language L is a set of ordered pairs of sentences of L that satisfy certain semantic rules that I will specify in a moment.

Rather than write '$\langle C, B\rangle \in \rightarrow$' (and '$\langle C, B\rangle \notin \rightarrow$'), I will usually write '$C \rightarrow B$' (and '$C \nrightarrow B$'). Define the notion of a *name-extension* of a set of pairs of sentences $\rightarrow$ on a language L to a set of pairs of sentences $\rightarrow^+$ on a language L^+, a name-extension of L, as follows.

DEFINITION 6. For any set $\rightarrow \subseteq S_L \times S_L$ for an $\mathcal{L}$-language L, if L^+ is a name-extension of L and $\rightarrow^+ \subseteq S_{L^+} \times S_{L^+}$, then $\rightarrow^+$ is a *name-extension* of $\rightarrow$ if and only if for all B, C in S_L, $C \rightarrow^+ B$ holds just in case $C \rightarrow B$ holds.

We now define the EQ-model classes of conditionals by extending the **ER** semantics for the Rational Consequence relations (of the previous paper) in the same way that TQ-model classes were defined by extending the truth-value semantics **TVA**.

DEFINITION 7. R is an *EQ-model class if and only if* for all $\rightarrow \in R$, $\rightarrow$ is a subset of pairs of sentence of an $\mathcal{L}$-language L that satisfies the following conditions:

$\rightarrow$ satisfies the rules of **ER** applied to sentences of L; i.e. for all A, B, C in S_L, $\rightarrow$ satisfies the following rules:

1) for some D and E in S_L, $E \nrightarrow D$;
2) $A \rightarrow A$;
3) if $(C\&B) \rightarrow A$, then $(B\&C) \rightarrow A$;
4.1) if $C \rightarrow (B\&A)$, then $C \rightarrow (A\&B)$;
4.2) if $C \rightarrow \neg(B\&A)$, then $C \rightarrow \neg(A\&B)$;
5.1) if $B \rightarrow \neg\neg A$, then $B \rightarrow A$;
5.2) if $B \rightarrow A$ and $B \rightarrow \neg A$, then $B \rightarrow C$;
6.1) $C \rightarrow B$ and $(C\&B) \rightarrow A$ *iff* $C \rightarrow (B\&A)$;
6.2) $C \rightarrow \neg B$ or $(C\&B) \rightarrow \neg A$ *iff* $C \rightarrow \neg(B\&A)$;

and for all sentences B in S_L and all formulas Fx in L containing only x free:

7) if $B \rightarrow (x)Fx$, then $B \rightarrow Fc$ for all names c in L;
8) *if* for each $\rightarrow^+ \in R$ such that $\rightarrow^+$ is a name-extension of $\rightarrow$ to L^+ (where L^+ is a name-extension of L), for every name c in L^+, $B \rightarrow^+ Fc$, *then* $B \rightarrow (x)Fx$.

The intuitive idea behind taking conditionals (in EQ-model classes) as semantic primitives (in place of truth-values) is that associated with each nonmonotonic conditional, $\rightarrow$, is a way of assigning meanings to sentences and a way of measuring classes of possible worlds; and, relative to its associated meanings and measures, a conditional assertion $C \rightarrow B$

holds *just in case* B is true in almost all possible worlds in which C is true. The semantic rules are intended as minimal constraints on any conditional that may bear this reading. In particular, for each conditional in **ER**, if $C \rightarrow A$, then either $C\&B \rightarrow A$ or $C \rightarrow \neg B$ (this follows primarily from rule 6.2). Thus, for a conditional $\rightarrow$ in **ER** (or, in an EQ-model class), if A is true in almost all worlds in which C is true, *then* A must also be true in the subset of C worlds in which B is true as well, *unless* B is false in almost all C worlds (in which case the $(C\&B)$ worlds may or may not almost all make A true, and may even almost all make A false). See my previous paper for more about this.

Rule 8 provides a treatment of quantifiers for EQ-model classes that is closely analogous to the treatment for TQ-model classes. When a sentence B *fails to support* the claim that everything has F (according to support relation $\rightarrow$), then there must be some possible way (consistent with how $\rightarrow$ is understood) of extending the support relation to a new name c (that, presumably, names some object) such that B *fails to support* Fc. An EQ-model class captures this idea by representing a class of possible name-extensions of a given conditional to broader languages.

Rule 8 is equivalent to the following rule for EQ-model classes:

8*) *if* $B \rightarrow \neg(x)Fx$ and $B \nrightarrow D$ (for some sentence D in L), *then* for some $\rightarrow^+ \in R$ such that $\rightarrow^+$ is a name-extension of $\rightarrow$ to L^+ (where L^+ is a name extension of L), for some name c in L^+, $B \nrightarrow^+ Fc$.

In terms of existential quantifiers, rule 8* says that when a sentence B *supports* the claim that $(\exists x)Gx$ (according to a support relation $\rightarrow$), then either B *supports* every sentence or else there must be some possible way in R of extending the support relation a new name c such that Gc is *possible* relative to B (i.e. B *fails to support* $\neg Gc$).

To see that 8* follows from 8, observe that if the consequent of 8* is false, then $B \rightarrow (x)Fx$ holds (by 8), so either $B \nrightarrow \neg(x)Fx$ or $B \rightarrow D$ for all D. Conversely, to obtain 8 from 8*, suppose a class R satisfies rules 1–7 and 8*, and suppose rule 8 fails, as follows: $C \nrightarrow (x)Gx$, but for every name-extension L^+ of L and every $\rightarrow^+ \in R$ a name-extension of $\rightarrow$ to L^+, $C \rightarrow^+ Gc$ for every name c in L^+. Then for every name-extension L^+ of L and every $\rightarrow^+ \in R$ a name-extension of $\rightarrow$ to L^+, $C\&\neg(x)Gx \rightarrow^+ Gc$ for every name c in L^+; and yet $C\&\neg(x)Gx \rightarrow \neg(x)Gx$. By 8*, $C\&\neg(x)Gx \rightarrow D$ for all D. Thus $C \rightarrow (x)Gx$ (provable from **ER** rules), which contradicts $C \nrightarrow (x)Gx$.

Most EQ-model classes contain more than a single conditional and *its* name-extensions. But such classes can always be subdivided into EQ-model subclasses that are each composed of a single conditional and its name-extensions. Clearly, the union of all the EQ-model classes is itself an EQ-model class – it is the class of all Rational Consequence relations for predicate logic.

DEFINITION 8. $\mathbf{EQ}$ is the union of all EQ-model classes defined on $\mathcal{L}$-languages.

The semantics for $\mathbf{EQ}$ provided by Definition 7 is completely autonomous from truth-value semantics. The possible nonmonotonic conditionals, representing defeasible support relations, are the semantic primitives. Indeed, the conditionals in $\mathbf{EQ}$ give rise to their own notions of logical truth and logical consequence, versions of these logical concepts that are native to $\mathbf{EQ}$'s semantics and autonomous from the semantics of truth-values. We may define the $\mathbf{EQ}$ versions of these logical concepts as follows.

DEFINITION 9. B is an *EQ-logical consequence* of a set of sentences Γ (abbreviated $\Gamma => B$) *iff* for all $\mathcal{L}$-languages L such that B and the sentences in Γ are in S_L and for all $\rightarrow \in \mathbf{EQ}$ defined on L, for each sentence D in S_L, if $D \rightarrow C$ for each $C \in \Gamma$, then $D \rightarrow B$.

B is an *EQ-logically true* (i.e. $=> B$) *iff* for all $\mathcal{L}$-languages L such that $B \in S_L$ and for all $\rightarrow \in \mathbf{EQ}$ defined on L, for each sentence $D \in S_L$, $D \rightarrow B$.

In cases where the set Γ contains a single sentence C and $\Gamma => B$, I will simply write '$C => B$'.

The EQ-logical truths might better be called *EQ-logical certainties*, and the EQ-logical consequence relation may more appropriately be called the *EQ-logically certain support* relation, or something of that sort. But it turns out that these $\mathbf{EQ}$ logical notions extensionally coincide with the corresponding classical notions, and the terminology anticipates this equivalence. The equivalence will be proved shortly.

Those who have read my previous paper will notice that the definition of EQ-logical consequence differs significantly from the definition of ER-logical consequence in the earlier paper: B is an ER-logical consequence of C *iff* for all $\rightarrow \in \mathbf{ER}$, $C \rightarrow B$. This previous definition does not directly extend to infinite sets of premises, which Definition 9 handles easily. However, there is a straightforward relationship between the definition of EQ-logical consequence and the kind of definition given for ER-logical consequence. I will describe this relationship precisely in Theorem 4, near the end of this section.

In the previous section we saw that each truth-value assignment in $\mathbf{TQ}$ has an ω-complete name-extension in $\mathbf{TQ}$. An analogous property applies to conditionals in $\mathbf{EQ}$.

DEFINITION 10. Let $\to$ be a set of pairs of sentences of an $\mathcal{L}$-language L (where $\to$ is not necessarily in **EQ**). $\to$ is called *explicit* just in case, for all formulas Fx in L with only x free and for all B in S_L, $B \to (x)Fx$ if and only if for all names c in L, $B \to Fc$.

Explicit conditionals in **EQ** are rather like ω-complete truth-value assignments. A sentence that claims that everything has F fails to be supported (by B) just in case for some name c, Fc is not supported (by B) either. Indeed, replacing all occurrences of the phrase 'fails to be supported' with the phrase 'is not true' in the previous sentence produces a description of ω-completeness. Each *explicit* conditionals $\to$ in EQ forms its own EQ-model class $\{\to\}$.

The next theorem shows that each conditional in EQ can be extended to an *explicit* conditional in EQ. The theorem also shows that if $\to$ is any set of pairs of sentence on an $\mathcal{L}$-language L that has an *explicit* name-extension satisfying rules 1–6 of the semantics for **EQ**, then $\to$ is in **EQ**. Not every EQ-model class contains an *explicit* conditional, but all EQ-model classes consist of sequences of name-extensions of conditionals that "approach"explicit conditionals in the limit.

THEOREM 2. *For any set of pairs of sentences $\to$ on an $\mathcal{L}$-language L, $\to \in$ **EQ** if and only if $\to$ has an explicit name-extension that satisfies the* **ER** *rules* (*rules* 1–6 *of Definition* 7).

Proof. (I) Suppose $\to'$ is an *explicit* name-extension of the set of pairs $\to$ and satisfies rules 1–6. Since $\to'$ is *explicit* it also satisfies rule 7. Clearly $\to$ also satisfies rules 1–7 since $\to'$ is a name-extension of $\to$. It is easy to verify that the set $\{\to, \to'\}$ is an EQ-model class.

(II) Suppose $\to \in$ **EQ** on a language L_0. $\to$ is in some EQ-model class, call it R. Let $\to$ be designated '$\to_0$'. I will show that R contains conditionals that compose a certain nested sequence $\to_0, \to_1, \to_2, \ldots$, of name-extensions of $\to_0$ (each a name-extension of the preceding ones) on languages $L_0, L_1, L_2, \ldots$ (each a name-extension of the preceding ones). The union of the conditionals in the sequence is an *explicit* name-extension of $\to$.

For each conditional in R, let Σ be a sequence of all pairs of formulas $\langle D, Fx\rangle$ in the language of the conditional such that D is a sentence and Fx has only one free variable. Rule 8 of Definition 7 guarantees that there is a sequence of conditionals in R beginning with $\to_0$ such that each conditional $\to_i$ bears the following relationship to the preceding conditionals in the sequence:

> for each $k < i$, if $\langle D, Fx\rangle$ is one of the first i members of Σ_k for language L_k and $D \nrightarrow_{i-1} (x)Fx$,

then for some name c in L_i, $D \nrightarrow_i Fc$.

Let the language L_∞ be the union of all languages L_i in this sequence, and let $\rightarrow_\infty$ be the union of all conditionals $\rightarrow_i$ in the sequence. Clearly $\rightarrow_\infty$ is a name-extension of $\rightarrow$ that satisfies rules 1–7 for EQ-model classes (or else $\rightarrow_\infty$ must violate one of these rules for some sentence pairs in some L_i; but $\rightarrow_i$ agrees with $\rightarrow_\infty$ on L_i, and $\rightarrow_i$ is in R). Furthermore, $\rightarrow_\infty$ is *explicit*. For, suppose $D \nrightarrow_\infty (x)Fx$; then for some $\rightarrow_j \in R,\ D \nrightarrow_j (x)Fx$ where $\langle D, Fx\rangle$ is the kth member of Σ_j; so for some $i \geqslant j+k,\ D \nrightarrow_i Fc$ for c in L_i. Hence $D \nrightarrow_\infty Fc$. Thus, $\rightarrow_\infty$ is an *explicit* name-extension of $\rightarrow$ that satisfies rules 1–7. □

The **EQ** notion of logical consequence coincides (extensionally) with the classical notion, as the next theorem establishes.

THEOREM 3. $\Gamma => B$ *if and only if* $\Gamma \vDash B$.

Proof. (I) Suppose $\Gamma \nvDash B$. Then there is an ω-complete truth-value assignment $v \in \mathbf{TQ}$ such that $\Gamma \cup \{\neg B\} \subseteq v$. Define $\rightarrow = \{\langle E, D\rangle \mid v[E \supset D]\}$. $\{\rightarrow\}$ is an EQ-model class (check Definition 7), so $\rightarrow \in \mathbf{EQ}$. And for any tautology $\mathbf{T}$ of sentential logic, $v[\mathbf{T} \supset C]$ for all $C \in \Gamma$, and not $v[\mathbf{T} \supset B]$. Thus, $\Gamma \not=> B$.

(II) Suppose $\Gamma \not=> B$. Then there is a conditional $\rightarrow$ in $\mathbf{EQ}$ on a language L for Γ and B, and a sentence D in L such that for all $C \in \Gamma$, $D \rightarrow C$ and $D \nrightarrow B$. Without loss of generality, we may suppose that $\rightarrow$ is *explicit*. The strategy now is to use a technique similar to a Henkin proof to build from $\rightarrow$ a truth-value assignment v in $\mathbf{TQ}$ on language L such that $v[C]$ for all $C \in \Gamma$ and $v[\neg B]$. Let $A_1, A_2, \ldots$ be an enumeration of the sentences of L, and let $c_1, c_2, \ldots$ be an enumeration of the names. If B is of form $(x)Fx$, let c_m be the first name in the enumeration such that $D\&\neg B \nrightarrow Fc_m$ (such a c_m exists, for: $\rightarrow$ is *explicit*; so, if $D\&\neg(x)Fx \rightarrow Fc$ for all c, then $D\&\neg(x)Fx \rightarrow (x)Fx$, so $D \rightarrow (x)Fx$ by **ER** rules). If B is of form $(x)Fx$, define $\rightarrow_0$ such that for all X and Y in S_L, $Y \rightarrow_0 X$ iff $((Y\&\neg B)\&\neg Fc_m) \rightarrow X$. If B is not of form $(x)Fx$, define $\rightarrow_0$ such that $Y \rightarrow_0 X$ iff $(Y\&\neg B) \rightarrow X$. The **ER** rules imply that for all sentences E, if $D \rightarrow E$, then $D \rightarrow_0 E$; also $D \rightarrow_0 \neg B$ and $D \nrightarrow_0 B$ (and $D \rightarrow_0 \neg Fc_m$ if B is of form $(x)Fx$). And $\rightarrow_0$ is clearly in $\mathbf{EQ}$. Now we extend $\rightarrow_0$ in a sequence of conditional $\rightarrow_i$ in a similar way by going through an enumeration of all sentences of L:

(1) If $D \rightarrow_{i-1} A_i$, then define $\rightarrow_i$ such that:

(1.1) $Y \rightarrow_i X$ iff $Y\&A_i \rightarrow_{i-1} X$, if A_i is not of form $\neg(x)Fx$;

(1.2) $Y \rightarrow_i X$ iff $((Y \& A_i) \& \neg Fc_k) \rightarrow_{i-1} X$, if A_i is of form $\neg(x)Fx$ and c_k is the first name in the enumeration such that $D \& A_i \nrightarrow_{i-1} Fc_k$.

(2) If $D \nrightarrow_{i-1} A_i$ then define $\rightarrow_i$ such that:

(2.1) $Y \rightarrow_i X$ iff $Y \& \neg A_i \rightarrow_{i-1} X$, if A_i is not of form $(x)Fx$;

(2.2) $Y \rightarrow_i X$ iff $((Y \& \neg A_i) \& \neg Fc_k) \rightarrow_{i-1} X$, if A_i is of form $(x)Fx$ and c_k is the first name in the enumeration such that $D \& \neg A_i \nrightarrow_{i-1} Fc_k$.

In either case $\rightarrow_i$ is in $\mathbf{EQ}$, and $D \rightarrow_i C$ for all C in Γ, and $D \rightarrow_i \neg B$ and $D \nrightarrow_i B$. Let $\rightarrow_\infty$ be the union of all of the $\rightarrow_i$. $\{\rightarrow_\infty\}$ is an EQ-model class since $\rightarrow_\infty$ is *explicit* (for, if $\rightarrow_\infty$ failed any of the $\mathbf{EQ}$ rules or failed to be *explicit*, then so must one of the $\rightarrow_i$). Define v as the set of all sentences E such that $D \rightarrow_\infty E$. Then v is an ω-complete member of $\mathbf{TQ}$. But $v[C]$ for all $C \in \Gamma$ and $v[\neg B]$. □

(For those familiar with my earlier paper – with the aid of Theorem 3 it is fairly easy to prove that $\mathbf{EQ}$ is a predicate logic version of the Rational Consequence relations described in the previous paper. That is, if the rules of the set of conditionals called $\mathbf{R}$ in the previous paper (the rules in Definitions 9, 11, and 12 of the previous paper) are applied to first-order sentences of $\mathcal{L}$-languages, and if the notion of logical consequence used in those rules is understood to be classical (objectual) logical consequence for predicate logic, then the resulting theory (call it $\mathbf{RQ}$) of nonmonotonic conditionals is equivalent to $\mathbf{EQ}$. The rules of $\mathbf{RQ}$ may easily be derived from those of $\mathbf{EQ}$ with the help of Theorem 3; so all conditionals in $\mathbf{EQ}$ are also in $\mathbf{RQ}$. Moreover, each conditional in $\mathbf{RQ}$ can be shown to have an *explicit* name-extension that satisfies the rules 1–6 of $\mathbf{EQ}$, so all conditionals in $\mathbf{RQ}$ belong to $\mathbf{EQ}$ (by Theorem 2).)

We now establish that the definition of EQ-logical consequence (Definition 9) is equivalent to an alternative characterization (which matches the form of the definition of ER-logical consequence offered in my previous paper).

THEOREM 4. $\Gamma => B$ *if and only if there is a finite subset* $\{C_1, \ldots, C_n\}$ *of* Γ *such that for all* $\rightarrow \in \mathbf{EQ}$ *on a language containing* B *and sentences in* $\{C_1, \ldots, C_n\}$, $(\cdots(C_1 \& C_2) \& \cdots \& C_n) \rightarrow B$.

Proof. First, Theorem 3 together with the compactness of classical logical consequence establishes that $\Gamma => B$ just in case there is a finite subset $\{C_1, \ldots, C_n\}$ on Γ such that $\{C_1, \ldots, C_n\} => B$. Notice too that $\{C_1, \ldots, C_n\} => B$ just in case $(C_1 \& \cdots \& C_n) => B$. (This follows from the **ER** rules, which suffice to prove that for each sentence

$D,\ D \to (C_1\&\cdots\&C_n)$ just in case for each $C_i \in \Gamma,\ D \to C_i$). Now we show that for any sentence $C,\ C => B$ just in case $C \to B$ holds for every $\to$ in $\mathbf{EQ}$ on languages containing C and B.

(I) Suppose $C \to B$ for all $\to \in \mathbf{EQ}$ on languages for C and B. Then $D\&C \to B$ for all $\to \in \mathbf{EQ}$ on the language of $C,\ D$, and B (for, if $D\&C \nrightarrow B$, then there is a conditional $\to' \in \mathbf{EQ}$ such that for all X and Y in the language of $\to$, $Y \to' Y$ iff $D\&X \to Y$; and $C \nrightarrow' B$). So, for each $\to \in \mathbf{EQ}$ and any D, if $D \to C$ (and $D\&C \to B$), then $D \to C\&B$; so $D \to B$. Consequently, $C => B$.

(II) Suppose $C => B$. Then for all $\to \in \mathbf{EQ}$ on a language containing C and B and for all D in this language, if $D \to C$, then $D \to B$. But $C \to C$ for all such conditionals $\to$, so $C \to B$ for all such $\to$. □

The truth-value semantics for $\mathbf{TQ}$ was primarily of interest for the aid it provided in making sense of the quantifier rules for the semantics of $\mathbf{EQ}$. And quantifier rules of the semantics for $\mathbf{PQ}$ (the **P**robabilistic semantics for **Q**uantifiers) presented in the next section is also rather similar to $\mathbf{TQ}$. However, the semantics for $\mathbf{EQ}$ is, of course, interesting in its own right. It extends one of the main systems of sentential nonmonotonic conditionals to predicate logic. These nonmonotonic conditionals constitute an autonomous semantic theory for predicate logic, a semantics of defeasible support (under interpretation) that is independent of the semantics of truth-values (under interpretation). The properties of conditionals in $\mathbf{EQ}$ will also shed light on the nature of the conditional probability functions that will make up $\mathbf{PQ}$. We will see that the conditionals in $\mathbf{EQ}$ are the probability 1 parts of the conditional probability functions in $\mathbf{PQ}$, thus extending the relationship between their respective sentential logic parts $\mathbf{ER}$ and $\mathbf{POPPERFN}$. Proof of the claim that the semantics for $\mathbf{PQ}$ captures the classical notion of logical consequence will, indeed, derive quite easily from the fact that $\mathbf{EQ}$-logical consequence coincides with the classical notion.

3. Conditional probabilities

In "Logic, Meaning, and Conceptual Role" Field (1977) extended the Popper Functions to first-order languages by extending the semantics of $\mathbf{POPPERFN}$ in much the same way that I built $\mathbf{EQ}$ from $\mathbf{ER}$ in the previous section. In this section I will provide a somewhat modified version of Field's probabilistic semantics, and show that it leads to the same logical consequence relation as the truth-value semantics for $\mathbf{TQ}$

and the nonmonotonic support-relation semantics for $\mathbf{EQ}$. The support-relation semantics $\mathbf{EQ}$ will turn out to be a subtheory of probabilistic semantics. In the next section I will explicitly compare Field's version of probabilistic semantics with the version developed in the present section.

Several researchers have developed probabilistic semantic theories for predicate logic by extending the Popper Functions. Harper's (1974) and Field's (1977) are the ground breaking works on probabilistic semantics for first-order languages. Harper first developed probabilistic semantics for a free quantificational logic (see his (1983) for a summary). Field (1977) independently constructed a probabilistic semantics for standard predicate logic. Leblanc conducted pioneering work on Popper functions, and has developed versions of probabilistic semantics that reflect the substitutional reading of quantifiers (see 1979, 1983a, b). Van Fraassen (1981) takes another interesting approach to probabilistic semantics for predicate logic. All of these approaches extend the Popper Functions to quantifiers by employing semantic rules that essentially say of each probability function P that it (or some name-extension of it) satisfies the following condition: for $c_1, c_2, \ldots$ a list of all names in the language of P, $P[(x)Fx \mid B] = \lim_n P[(Fc_1\&\cdots\&Fc_n) \mid B]$. The semantics I will specify does not appeal to this kind of condition. Rather I will employ semantic rules for quantifiers that are precise analogues of the simpler quantifier rules of $\mathbf{EQ}$.

In order to extend the Popper Functions to a semantics for predicate logic, I will first define the notion of a *PQ-model class*. PQ-model classes provide a means of representing the import of quantifiers; they are the probabilistic analogues of the model classes for $\mathbf{TQ}$ and for $\mathbf{EQ}$. The main idea is that whenever $(x)Fx$ is uncertain given B (i.e. whenever $P[(x)Fx \mid B] < 1$ for a function P in model class R), there must be a name-extension of the language that contains at least one name c such that Fc is uncertain given B (i.e. there must be a P^+, a name-extension of P, in R such that for some name c, $P^+[Fc \mid B] < 1$).

DEFINITION 11. R is a *PQ-model class if and only if* for all $P \in R,\ P$ is a function from all pairs of sentence of an $\mathcal{L}$-language L into the real numbers in the interval [0,1], and P satisfies the following conditions:

P satisfies the rules of $\mathbf{POPPERFN}$ applied to sentences of L; i.e. for all $A,\ B,\ C$ in S_L, P satisfies the following rules:

1) for some D and E in S_L, $P[D \mid E] \neq 1$;
2) $P[A \mid A] = 1$;
3) $P[A \mid C\&B] = P[A \mid B\&C]$;
4) $P[B\&A \mid C] = P[A\&B \mid C]$;

5) $P[A \mid B] + P[\neg A \mid B] = 1$ or $P[C \mid B] = 1$;
6) $P[A \& B \mid C] = P[A \mid B \& C] \times P[B \mid C]$;

for all sentences B in S_L and all formulas Fx in L containing only x free:

7) if $P[(x)Fx \mid B] = 1$, then $P[Fc \mid B] = 1$ for all names c in L;
8) *if* for each $P^+ \in R$ such that P^+ is a name-extension of P to L^+ (i.e. where L^+ is a name-extension of L, and P agrees with P^+ on sentences of L), for every name c in L^+, $P^+[Fc \mid B] = 1$, *then* $P[(x)Fx \mid B] = 1$.

The union of all PQ-model classes is itself a PQ-model class; I call this class $\mathbf{PQ}$, because it derives from a $\mathbf{P}$robabilistic semantics for $\mathbf{Q}$uantifiers.

DEFINITION 12. $\mathbf{PQ}$ is the union of all PQ-model classes defined on $\mathcal{L}$-languages.

In the previous paper we saw that the nonmonotonic conditionals in $\mathbf{ER}$ are just the probability 1 parts of the conditional probability functions in $\mathbf{POPPERFN}$. A similar relationship holds between $\mathbf{EQ}$ and $\mathbf{PQ}$. To establish this claim we first define the set of conditionals $\mathbf{PQ}[1]$ that correspond to the probability 1 parts of the conditional probability functions in $\mathbf{PQ}$ as follows:

DEFINITION 13. For each $\mathcal{L}$-language L, $\rightarrow \in \mathbf{PQ}[1]$ on L if and only if for some $P \in \mathbf{PQ}$ on L, $\rightarrow = \{\langle C, B\rangle \mid P[B \mid C] = 1\}$.

In the previous paper I established that for sentential logic $\mathbf{ER} = \mathbf{POPPERFN}[1]$ (Theorems 4 and 5 of the previous paper). The next theorem establishes the corresponding result for predicate logic.

THEOREM 5. $\mathbf{PQ}[1] = \mathbf{EQ}$.

Proof. (I) First let's establish that $\mathbf{PQ}[1] \subseteq \mathbf{EQ}$. We only need show that $\mathbf{PQ}[1]$ is an EQ-model class. Clearly each $\rightarrow$ in $\mathbf{PQ}[1]$ satisfies rules 1–7 of an EQ-model class. To see that rule 8 is satisfied, suppose $\rightarrow$ is in $\mathbf{PQ}[1]$. Then there is some function P in $\mathbf{PQ}$ such that $\rightarrow = \{\langle C, D\rangle \mid P[D \mid C] = 1\}$. Now, suppose that for each $\rightarrow^+ \in \mathbf{PQ}[1]$ such that $\rightarrow^+$ is a name-extension of $\rightarrow$ to L^+, for every name c in L^+, $B \rightarrow^+ Fc$. Then, for each function $P^+ \in \mathbf{PQ}$ such that P^+ is a name-extension of P to L^+, for every name c in L^+, $P^+[Fc \mid B] = 1$. Then $P[(x)Fx \mid B] = 1$. Hence, $B \rightarrow (x)Fx$.

(II) Now let's see that $\mathbf{EQ} \subseteq \mathbf{PQ}[1]$. Suppose $\rightarrow \in \mathbf{EQ}$ on language L. Let $\rightarrow^*$ be an *explicit* name-extension of $\rightarrow$. For the moment think of each sentence that begins with a universal quantifiers in the language L^* of $\rightarrow^*$ as a sentence letter; then Theorems 4 and 5 of the previous paper imply that there is a probability function P^* satisfying rules 1–6 of $\mathbf{PQ}$ such that for all X and Y of L^*, $P^*[X \mid Y] = 1$ just when $Y \rightarrow^* X$. P^* satisfies rule 7 for PQ-model classes, since: $P^*[(x)Fx \mid B] = 1$ *only if* $B \rightarrow^* (x)Fx$ *only if* for each c in L^*, $B \rightarrow^* Fc$ *only if* for each c in L^*, $P^*[Fc \mid B] = 1$. $\{P^*\}$ is a PQ-model class since (given any B and Fx in L^*): for all c in L^*, $P^*[Fc \mid B] = 1$ *only if* for all c in L^*, $B \rightarrow^* Fc$ *only if* $B \rightarrow^* (x)Fx$ (since $\rightarrow^*$ is *explicit*) *only if* $P^*[(x)Fx \mid B] = 1$. Let P be P^* restricted to the language L. $\{P, P^*\}$ is also a PQ-model class, so $P \in \mathbf{PQ}$. But $\rightarrow$ is the restriction of $\rightarrow^*$ to L, and $\rightarrow = \{\langle Y, X\rangle \mid P[X \mid Y] = 1\}$. Thus $\rightarrow \in \mathbf{PQ}[1]$. $\square$

We can define a notion of logical consequence in terms of the probability functions in $\mathbf{PQ}$. PQ-logical consequence will turn out to be extensionally equivalent to the classical notion.

DEFINITION 14. $\Gamma \Rightarrow B$ (i.e. B is a *PQ-logical consequence* of a set of sentences Γ) *iff* for all $\mathcal{L}$-languages L such that B and the sentences in Γ are in S_L and for all $P \in \mathbf{PQ}$ defined on L, for each sentence D in S_L, if $P[C \mid D] = 1$ for each $C \in \Gamma$, then $P[B \mid D] = 1$.

B is a *PQ-logical truth* (i.e. $\Rightarrow B$) *iff* for all $\mathcal{L}$-languages L such that $B \in S_L$ and all $P \in \mathbf{PQ}$ defined on L, for each sentence $D \in S_L$, $P[B \mid D] = 1$.

The PQ-logical "truths" might better be called *probabilistic-logical certainties* and the PQ-logical consequence relation may more appropriately be called the *probabilistic-logically certain support* relation (or something of that kind). I employ the terms 'logical truth' and 'logical consequence' in order to alert the reader to the fact that these $\mathbf{PQ}$ logical properties turn out to extensionally coincide with the corresponding classical logical notions.

From the fact that EQ-logical consequence coincides with classical logical consequence (Theorem 3 of the previous section) it is fairly straightforward to prove that the probabilistic notion of logical consequence is extensionally equivalent to the classical notion.

THEOREM 6. $\Gamma \Rightarrow B$ *if and only if* $\Gamma \vDash B$.

Proof. (I) Suppose $\Gamma \nvDash B$. Then there is an ω-complete truth-value assignment $v \in \mathbf{TQ}$ such that $\Gamma \cup \{\neg B\} \subseteq v$. Define a function P

on the sentences of L as follows: $P[X \mid Y] = 1$ if $v[Y \supset X]$, and $P[X \mid Y] = 0$ otherwise. $\{P\}$ satisfies the rules for a PQ-model class. And for any tautology $\mathbf{T}$ of sentential logic, $v[\mathbf{T} \supset C]$ for all $C \in \Gamma$, and not $v[\mathbf{T} \supset B]$. Thus $\Gamma \not\Rightarrow B$.

(II) Suppose $\Gamma \not\Rightarrow B$. There is a function P in $\mathbf{PQ}$ on a language L containing the sentences in Γ and sentences B and D such that for all $C \in \Gamma$, $P[C \mid D] = 1$ and $P[B \mid D] < 1$. From Theorem 5 it follows that there is a conditional $\rightarrow \in \mathbf{EQ}$ such that $D \rightarrow C$ for all C in Γ and $D \not\rightarrow B$. So, $\Gamma \neq> B$. Then, by Theorem 3, $\Gamma \nvDash B$. □

There are several equivalent ways to characterize the PQ-logical consequence relation. Theorems 7 and 8 will establish two natural alternatives. Theorem 7 shows that the definition of logical consequence for $\mathbf{POPPERFN}$ in the previous paper works for $\mathbf{PQ}$ as well.

THEOREM 7. $\Gamma \Rightarrow B$ *if and only if there is a finite subset* $\{C_1, \ldots, C_n\}$ *of* Γ *such that for all* $P \in \mathbf{PQ}$ *on languages containing* B *and the sentences in* $\{C_1, \ldots, C_n\}$, $P[B \mid (C_1 \& \cdots \& C_n)] = 1$.

Proof. $\Gamma \Rightarrow B$ *iff* $\Gamma \vDash B$ *iff* $\Gamma => B$ *iff* there is a finite subset $\{C_1, \ldots, C_n\}$ of Γ such that for all $\rightarrow \in \mathbf{EQ} = \mathbf{PQ}[1]$ on languages containing B and the sentences in $\{C_1, \ldots, C_n\}$, $(C_1 \& \cdots \& C_n) \rightarrow B$. This last claim holds *iff* there is a finite subset $\{C_1, \ldots, C_n\}$ of Γ such that for all $P \in \mathbf{PQ}$ on languages containing B and the sentences in $\{C_1, \ldots, C_n\}$, $P[(C_1 \& \cdots \& C_n) \mid B] = 1$. □

Field (1977) defines probabilities on sets of sentences, including infinite sets. He employs probabilities of sets of sentences in another probabilistic characterization of logical consequence. Field's definition of probabilities on sets goes like this:

DEFINITION 15. Let Γ be a countable set of sentences of $\mathcal{L}$-language L and let C be a sentence of L; let $P \in \mathbf{PQ}$ be defined on L. If Γ is finite and $\Gamma = \{C_1, \ldots, C_n\}$, then define $P[\Gamma \mid C] = P[(C_1 \& \cdots \& C_n) \mid C]$. If Γ is countably infinite and $C_1, C_2, \ldots C_n, \ldots$ is an enumeration of the members of Γ, define $P[\Gamma \mid C] = \lim_n P[(C_1 \& \cdots \& C_n) \mid C]$.

The functions P are clearly well defined on finite sets Γ since the rules of $\mathbf{POPPERFN}$ imply that the value of the probability of a string of conjuncts does not vary with order. The limit must exist because the values of $P[(C_1 \& \cdots \& C_n) \mid C]$ as n increases are monotonically decreasing and bounded below by 0; and it is axiomatic of the real numbers that every set of real numbers with a lower bound has a greatest lower bound (the limit).

When Γ is an infinite set of sentences there is no natural definition of conditional probabilities $P[B \mid \Gamma]$ as limits of finite subsets. For, the values of $P[B \mid (C_1 \& \cdots \& C_n)]$ can swing up and down as n increases, and they may not converge to any limit. Even when a limit does exist, its value may depend on the order in which the C_i are enumerated. However, the definition of $P[\Gamma \mid C]$ for infinite Γ does not encounter these difficulties since it is monotonically descreasing and bounded below (by 0).

Field calls the inference from a set of sentences Γ to a sentence B *probabilistically valid* just in case for every function P (in $\mathbf{PQ}$), $P[B \mid C] \geqslant P[\Gamma \mid C]$ for every sentence C in the language of P. The next theorem establishes that Field's characterization of *probabilistically valid* inferences is equivalent to the notion of PQ-logical consequence specified in Definition 14.

THEOREM 8. $\Gamma \Rightarrow B$ *if and only if for all* $P \in \mathbf{PQ}$ *on a language* L *containing* B *and the sentences in* Γ, *for all* C *in* S_L, $P[B \mid C] \geqslant P[\Gamma \mid C]$.

Proof. (I) Suppose $\Gamma \Rightarrow B$. By Theorem 7, for some $\{C_1, \ldots, C_n\}$ a finite subset of Γ, for all $P \in \mathbf{PQ}$ on a language of B and $\{C_1, \ldots, C_n\}$, $P[B \mid (C_1 \& \cdots \& C_n)] = 1$. For brevity, let '$D$' represent the sentence $(C_1 \& \cdots \& C_n)$. Then, for any such P and all C in the language for P, $P[B \mid D \& C] = 1$ (otherwise the following function P' would be in $\mathbf{PQ}$ and $P'[B \mid D] < 1$: $P'[X \mid Y] = P[X \mid Y \& C]$ for all X, Y in the language of P). So $P[B \mid C] \geqslant P[D \mid B \& C] \times P[B \mid C] = P[D \& B \mid C] = P[B \mid D \& C] \times P[D \mid C] = P[D \mid C] \geqslant P[\Gamma \mid C]$.

(II) Suppose for all $P \in \mathbf{PQ}$ on languages containing B and the sentences in Γ, and for all C in that language, $P[B \mid C] \geqslant P[\Gamma \mid C]$. Let P' be any member of $\mathbf{PQ}$ on a language for B and Γ such that for all C in the language, $P'[C_i \mid C] = 1$ for all C_i in Γ. Then $P'[\Gamma \mid C] = 1$; so $P'[B \mid C] = 1$. □

In the previous paper I provided a more classical set of *axioms* for Popper Functions, and I called the set of probability functions that satisfy these axioms $\mathbf{CONDPROB}$ (Definition 5 of that paper). These axioms define conditional probability functions on a sentential language in terms of more classical looking axioms, axioms that *employ* the classical notion of logical consequence. We may extend the rules for $\mathbf{CONDPROB}$ to all first-order sentences of $\mathcal{L}$-languages simply by taking the notion of logical consequence employed in its rules to be classical (objectual) logical consequence for predicate logic. Call the resulting theory '$\mathbf{CPQ}$' (for $\mathbf{CONDPROB}$ with $\mathbf{Q}$uantifiers).

DEFINITION 16. $P \in \mathbf{CPQ}$ *iff* P is a function from all pairs of sentences of an $\mathcal{L}$-language L into $[0, 1]$ such that:

1) for some E and G in S_L, $P[G \mid E] \neq 1$; and for all $A,\ B,\ C,\ D$ in S_L,
2) if $\vDash C \equiv B$, then $P[A \mid B] = P[A \mid C]$;
3) if $C \vDash A$, then $P[A \mid C] = 1$;
4) if $C \vDash \neg(A\&B)$, then either $P[A \vee B \mid C] = P[A \mid C] + P[B \mid C]$ or $P[D \mid C] = 1$;
5) $P[A\&B \mid C] = P[A \mid B\&C] \times P[B \mid C]$.

The rules of $\mathbf{CPQ}$ are easily derived from those of $\mathbf{PQ}$ with the help of Theorem 6; so all conditionals in $\mathbf{PQ}$ are also in $\mathbf{CPQ}$ – i.e. $\mathbf{PQ} \subseteq \mathbf{CPQ}$.

Is it also the case that $\mathbf{CPQ} \subseteq \mathbf{PQ}$? Each function in $\mathbf{CPQ}$ is easily shown to satisfy rules 1–7 of PQ-model classes. But rule 8 for $\mathbf{PQ}$ is harder to verify for functions in $\mathbf{CPQ}$. The issue can be clarified by introducing the notion of an *explicit* probability function, in analogy with the notion of an *explicit* conditional as specified in Definition 10.

DEFINITION 17. Let P be a function from all pairs of sentences of an $\mathcal{L}$-language L into the interval $[0, 1]$ (where P is not necessarily in $\mathbf{CPQ}$ or in $\mathbf{PQ}$). P is called *explicit* just in case for all formulas Fx in L with only x free and for all sentences B in S_L, if $P[Fc \mid B] = 1$ for all c in L, then $P[(x)Fx \mid B] = 1$.

Corresponding to Theorem 2 – which states the relationship between explicit conditionals and members of $\mathbf{EQ}$ – we have the following theorem for $\mathbf{PQ}$.

THEOREM 9. *For any function P from all pairs of sentences of an $\mathcal{L}$-language L into* $[0, 1]$*, $P \in$* **PQ** *if and only if $\rightarrow$ has an explicit name-extension that satisfies the* **POPPERFN** *rules* (1–6 *of Definition* 11)*.*

Theorem 9 can be proved by simply rewriting the proof of Theorem 2 with 'functions from $\mathcal{L}$-languages into $[0, 1]$' in place of 'sets of pairs of sentences', and with 'PQ' in place of 'EQ'.

Now we can specify a subset of $\mathbf{CPQ}$ that is bound to coincide with $\mathbf{PQ}$.

DEFINITION 18. Define $\mathbf{CPQ}^*$ as the set of all functions in $\mathbf{CPQ}$ that have *explicit* name-extensions in $\mathbf{CPQ}$. Each *explicit* member of $\mathbf{CPQ}$ is an *explicit* name-extension of itself, and so must also belong to $\mathbf{CPQ}^*$.

Since $\mathbf{PQ} \subseteq \mathbf{CPQ}$, Theorem 9 guarantees that $\mathbf{PQ} \subseteq \mathbf{CPQ}^*$. And $\mathbf{CPQ}^*$ is a PQ-model class (by the rules of Definition 11), so $\mathbf{CPQ}^* \subseteq \mathbf{PQ}$. In sum we have established the following theorem.

THEOREM 10. $\mathbf{PQ} = \mathbf{CPQ}^* \subseteq \mathbf{CPQ}$.

I strongly suspect that every probability function in $\mathbf{CPQ}$ has an *explicit* name-extension in $\mathbf{CPQ}$, but I have not worked out a satisfactory proof of this conjecture. (The idea, though, is to represent each probability function in $\mathbf{CPQ}$ by a monadic "classical" probability function on the non-standard reals – see the appendix of Lehmann and Magidor (1992) and the paper by McGee (1994) – and then to adapt a proof by Gaifman (1964, Theorem 2) to these monadic probability functions. I will say a bit more about this at the end of the next section.) If this conjecture proves correct, then indeed $\mathbf{PQ} = \mathbf{CPQ}^* = \mathbf{CPQ}$.

4. **PQ** AND THE *REASONABLE* PROBABILITY FUNCTIONS

Field's semantic rules for quantifiers in his (1977) differ in some important respects from rules 7 and 8 for PQ-model classes. In this section I will recount Field's version of probabilistic semantics and explore its relationship to the semantics of $\mathbf{PQ}$.

Field calls the counterparts of *PQ-model classes* in his version of probabilistic semantics *reasonability classes*. He defines them as follows:

DEFINITION 19. R is called a *reasonability class if and only if* for all $P \in R,\ P$ is a function from all pairs of sentence of an $\mathcal{L}$-language L into the real numbers in the interval $[0, 1]$, and P satisfies the following conditions:

> P satisfies the rules of **POPPERFN** applied to sentences of L; for all sentences B in S_L and all formulas Fx in L containing only x free:
>
> 7) $P[(Fc_1\& \cdots \& Fc_n) \mid B] \geqslant P[(x)Fx \mid B]$ for all names $c_1, \ldots,$ c_n in L;
> 8) for each real number r, *if* for each $P^+ \in R$ such that P^+ is a name-extension of P to L^+, for every list of names $c_1, \ldots, c_n$ in L^+, $P^+[(Fc_1\& \cdots \& Fc_n) \mid B] > r$, *then* $P[(x)Fx \mid B] \geqslant r$.

P is called *reasonable* just in case it is a member of some *reasonability class* on a $\mathcal{L}$-language; call the union of all *reasonability classes* **FPQ** (Field's version of **PQ**).

Definition 19 differs only insignificantly from the way Field defines *reasonability classes*. Field employs existential quantifiers and disjunctions rather than universal quantifiers and conjunctions. If we contrapose rule 8, substitute '$\neg Gx$' for 'Fx', use the fact that $P[\neg X \mid B] = 1 - P[X \mid B]$ (or else for all $D,\ P[D \mid B] = 1$), and take the definition of '$(Gc_1 \vee \cdots \vee Gc_n)$' to be '$\neg(\neg Gc_1 \& \cdots \& \neg Gc_n)$', then we recover the rule in the form in which Field states it:

> for each real number $s\ (= 1 - r)$, *if* $P[(\exists x)Gx \mid B] > s$, *then* there is a $P^+ \in R$ such that P^+ is a name-extension of P to L^+, and there is a list of names $c_1, \ldots, c_n$ in L^+, such that $P^+[(Gc_1 \vee \cdots \vee Gc_n) \mid B] \geqslant s$.

Rule 8 of Definition 19 captures the idea that if $P[(x)Fx \mid B] < r$ (for P in *reasonability class* R), then there must be some way to introduce a conjunction of assertions about named things that *approximates the uncertainty* of the universally quantified assertion as closely as one wishes – i.e. there must be some P^+ in R with names $c_1, \ldots, c_n$ in its language such that $P[(x)Fx \mid B] \leqslant P[(Fc_1 \& \cdots \& Fc_n) \mid B] \leqslant r$.

Rules 7 of the definitions of PQ-model classes and of *reasonability classes* are equivalent in the context of the first six rules for Popper Functions. For, it follows immediately from rule 7 of Definition 19 (for $\mathbf{FPQ}$) that if $P[(x)Fx \mid B] = 1$ then $P[Fc \mid B] = 1$ for all c. Conversely, suppose that $P[(Fc_1 \& \cdots \& Fc_n) \mid B] < P[(x)Fx \mid B]$ but that rule 7 of Definition 11 (for $\mathbf{PQ}$) holds. Then observe: $P[(x)Fx \mid B] > P[(x)Fx \& (Fc_1 \& \cdots \& Fc_n) \mid B] = P[(x)Fx \mid B] \times P[(Fc_1 \& \cdots \& Fc_n) \mid (x)Fx \& B]$; so $1 > P[(Fc_1 \& \cdots \& Fc_n) \mid (x)Fx \& B] = 1$ (since $P[(x)Fx \mid (x)Fx \& B] = 1$); a contradiction. So the two versions of rule 7 are interchangeable in Definition 11 and 19. The only substantial difference between the definition of *reasonability classes* and the definition of PQ-model arises in their versions of rule 8.

Every *reasonability class* is a PQ-model class since *reasonability classes* satisfy rule 8 of Definition 11. To see this, suppose R is a *reasonability class*, and suppose that for every $P^+ \in R$, a name-extension of P to L^+, and for all c in L^+, $P^+[Fc \mid B] = 1$; then for all such P^+ and all names $c_1, \ldots, c_n$ in L^+, $P^+[(Fc_1 \& \cdots \& Fc_n) \mid B] = 1$; so for all $r < 1$, for all such P^+ and all names $c_1, \ldots, c_n$ in L^+, $P^+[(Fc_1 \& \cdots \& Fc_n) \mid B] > r$; then for all $r < 1$, $P[(x)Fx \mid B] \geqslant r$; thus $P[(x)Fx \mid B] = 1$. Since **FPQ** is itself a *reasonability class*, it follows that $\mathbf{FPQ} \subseteq \mathbf{PQ}$.

Let's say that a sentence B is an *FPQ-logical consequence* of a set of sentences Γ just in case for all $P \in \mathbf{FPQ}$ on the language of Γ and B, for every D in the language of P, if $P[C \mid D] = 1$ for all C in Γ, then $P[B \mid D] = 1$. We can now immediately prove the following theorem.

THEOREM 11. *B is an FPQ-logical consequence of Γ if and only if $\Gamma \vDash B$.*

Proof. (I) Suppose that $\Gamma \nvDash B$. Then an ω-complete truth-value assignment that makes all sentences in Γ true and B false can provide a *reasonability class*, just as in part I of the proof of Theorem 6.

(II) Suppose $\Gamma \vDash B$, then (since $\Gamma \Rightarrow B$ and $\mathbf{FPQ} \subseteq \mathbf{PQ}$) B is an FPQ-logical consequence of Γ. □

Field, of course, did not have $\mathbf{PQ}$ or $\mathbf{EQ}$ to work with, so his proof of Theorem 11 takes more work. In order to prove his version of Theorem 11 Field introduces the notion of a *saturated* Popper Function. Saturated Popper Functions are similar to ω-complete truth-value assignments, and similar to *explicit* EQ-models.

DEFINITION 20. Let P be a function from all pairs of sentences of an $\mathcal{L}$-language L into the interval $[0, 1]$ (where P is not necessarily in $\mathbf{FPQ}$ or $\mathbf{PQ}$). P is called *saturated* just in case for all formulas Fx in L with only x free and for all sentences B in S_L, $\lim_n P[(Fc_1 \& \cdots \& Fc_n) \mid B] = P[(x)Fx \mid B]$, where $c_1, c_2, \ldots$ is a list of all the names in L. If there are only m names in L, the limit is understood to be $P[(Fc_1 \& \cdots \& Fc_m) \mid B]$.

Just as every PQ-model has an *explicit* name-extension, it turns out that every *reasonable* Popper Function has a *saturated* name-extension. Indeed **FPQ** consists of precisely those functions that have saturated name-extensions that satisfy the rules of **POPPERFN**. This result will prove useful; it was first proved in (Field, 1977).

THEOREM 12. *For any function P from pairs of sentences on an $\mathcal{L}$-langauge L into the real numbers in the interval $[0, 1]$, $P \in$ **FPQ** if and only if P has a saturated name-extension that satisfies the* **POPPERFN** *rules (i.e. rules* 1–6 *of Definition* 11).

Proof. (I) Suppose P' is a saturated name-extension of the function P and satisfies rules 1–6. Since P' is saturated it also satisfies rule 7. (If $P'[(Fc_1 \& \cdots \& Fc_n) \mid C] < P'[(x)Fx \mid C]$ for some names $c_1, \ldots, c_n$ in L, then since the probability of the conjunction cannot increase with additional conjuncts (rule 6), convergence to $P'[(x)Fx \mid C]$ would fail.) Clearly P also satisfies rules 1–7 since P' is a name-extension of P. It is easy to verify that the set $\{P, P'\}$ is a PQ-model class.

(II) Suppose $P \in \mathbf{PQ}$ on a language L_0. P is in some PQ-model class R. Let's designate P as 'P_0'. I will show that R contains probability functions that compose a certain nested sequence $P_0, P_1, P_2, \ldots,$ of name-extensions of P_0 (each a name-extension of the preceding ones) on languages $L_0, L_1, L_2, \ldots$ (each a name-extension of the preceding ones). Think of each function P_k as a set of triples: where $\langle B, C, r\rangle \in P_k$ just when $P_k[B \mid C] = r$. The union of the sequence of probability functions will be a saturated name-extension of P.

For each language L of a conditional in R, let Σ be a sequence of all pairs of formulas $\langle D, Fx\rangle$ from L such that D is a sentence and Fx has only one free variable; let σ be a sequence of all names of L. Rule 8 of Definition 11 guarantees that there is a sequence of functions in R beginning with P_0 such that each function P_i bears the following relationship to the preceding functions in the sequence:

> for each $k < i$, if $\langle D, Fx\rangle$ is one of the first i members of Σ_k for languages L_k, then for some initial sequence $c_1, \ldots, c_n$ in σ_i, $P_{i-1}[(x)Fx \mid D] \leqslant P_i[(Fc_1\&\cdots\&Fc_n) \mid D] \leqslant P_{i-1}[(x)Fx \mid D] + (1/2)^i$.

Rule 8 (together with 7) guarantees that such a P_i exists in R since for each such $\langle D, Fx\rangle$, $P_{i-1}[(x)Fx \mid D] < P_{i-1}[(x)Fx \mid D] + (1/2)^i$.

Let the language L_∞ be the union of all languages L_i in this sequence, and let P_∞ be the union of the P_i in the sequence. Clearly P_∞ is a name-extension of P that satisfies rules 1–7 for PQ-model classes (or else P_∞ must violate one of these rules for some sentence pairs in some L_i; but P_i agrees with P_∞ on L_i, and P_i is in R).

The following argument shows that P_∞ is saturated. Let D and Fx be in L_∞ and suppose, for a reductio, that $\lim_n P_\infty[(Fc_1\&\cdots\&Fc_n) \mid D] \neq P_\infty[(x)Fx \mid D]$. Then for some r in $[0,1]$, $P_\infty[(Fc_1\&\cdots\&Fc_n) \mid D] > r > P_\infty[(x)Fx \mid D]$ for all names in L_∞. But Fx and D belong to some language L_j; and for some k, $\langle D, Fx\rangle$ in the kth member in Σ_j. So for each $m > j+k$, $P_j[(x)Fx \mid D] < r < P_m[(Fb_1\&\cdots\&Fb_h) \mid D] \leqslant P_j[(x)Fx \mid D] + (1/2)^m$ for the b_i in an initial segment of σ_m. But for m large enough, $P_j[(x)Fx \mid D] + (1/2)^m < r$. So there must be some initial segment $c_1, \ldots, c_n$ of σ_∞ (which contains $b_1, \ldots, b_h$ as a subsequence) such that $P_\infty[(Fc_1\&\cdots\&Fc_n) \mid D] < r$, contradiction.

Therefore, P_∞ is a saturated name-extension of P that satisfies rules 1–7. □

Earlier we saw that $\mathbf{FPQ} \subseteq \mathbf{PQ}$, and we established in the previous section that $\mathbf{PQ} = \mathbf{CPQ}^* \subseteq \mathbf{CPQ}$. In a manner similar to the specification of $\mathbf{CPQ}^*$ in the previous section, we can specify a subset of $\mathbf{CPQ}$ that is bound to coincide with $\mathbf{FPQ}$.

DEFINITION 21. Define $\mathbf{CPQ}^{**}$ as the set of functions in $\mathbf{CPQ}$ that have *saturated* name-extensions in $\mathbf{CPQ}$. Notice that each *saturated* member of $\mathbf{CPQ}$ is a *saturated* name-extension of itself, and so must also belong to $\mathbf{CPQ}^{**}$.

Since $\mathbf{FPQ} \subseteq \mathbf{CPQ}$, Theorem 12 guarantees that $\mathbf{FPQ} \subseteq \mathbf{CPQ}^{**}$. And $\mathbf{CPQ}^{**}$ is a *reasonability* class (by the rules of Definition 19), so $\mathbf{CPQ}^{**} \subseteq \mathbf{FPQ}$. Thus we have established the following theorem.

THEOREM 13. $\mathbf{FPQ} = \mathbf{CPQ}^{**} \subseteq \mathbf{PQ} = \mathbf{CPQ}^* \subseteq \mathbf{CPQ}$.

I strongly suspect that $\mathbf{FPQ} = \mathbf{CPQ}$, but I have not yet worked out a satisfactory proof of this conjecture. (Indeed the same reason cited in the previous section for thinking that $\mathbf{CPQ}^* = \mathbf{CPQ}$ should apply equally to show that $\mathbf{CPQ}^{**} = \mathbf{CPQ}$ – represent each probability function in $\mathbf{CPQ}$ by a monadic "classical" probability function on the non-standard reals; then adapt the proof in Gaifman (1964, Theorem 2) to these monadic probability functions. Gaifman's Theorem 2 shows that every classical monadic probability function on a fist-order language can be extended to a *saturated* monadic probability function.) If this conjecture proves correct, then $\mathbf{FPQ} = \mathbf{PQ} = \mathbf{CPQ}$ and Field's version of probabilistic semantics turns out to specify precisely the same of probability functions as the semantics for $\mathbf{PQ}$.

5. Closing remarks

I will conclude this paper with a few remarks about how the semantic theories for nonmonotonic conditionals and conditional probabilities may be furhter expanded. The expansion of the semantics for $\mathbf{TQ}$, $\mathbf{EQ}$, and $\mathbf{PQ}$ to systems that treat identity as a logical constant is straightforward, and a similar approach will permit any first-order theory to be added as "part of the logic".

First, we can extend the semantic theories to theories with identity as follows:

DEFINITION 22. Suppose each $\mathcal{L}$-language has a special binary relation symbol '='. Define $\mathbf{TQ}^=$, $\mathbf{EQ}^=$, and $\mathbf{PQ}^=$ as follows:

$v \in \mathbf{TQ}^=$ *iff* $v \in \mathbf{TQ}$ on an $\mathcal{L}$-language L and

9) $v[c = c]$ for all names c in L;
10) for all names b and c and all formulas Fx with only x free in language L, if $v[b = c]$ and $v[Fb]$, then $v[Fc]$.

$\rightarrow \in \mathbf{EQ}^{=}$ *iff* $\rightarrow \in \mathbf{EQ}$ on an $\mathcal{L}$-language L and

9) $B \rightarrow c = c$ for all sentences B and names c in L;
10) for all names b and c, all sentences B and all formulas Fx with only x free in language L, if $B \rightarrow b = c$ and $B \rightarrow Fb$, then $B \rightarrow Fc$.

$P \in \mathbf{PQ}^{=}$ *iff* $P \in \mathbf{PQ}$ on an $\mathcal{L}$-language L and

9) $P[c = c \mid B] = 1$ for all sentences B and names c in L;
10) for all names b and c, all sentences B and all formulas Fx with only x free in language L, if $P[b = c \mid B] = 1$ and $P[Fb \mid B] = 1$, then $P[Fc \mid B] = 1$.

It should be noted that rule 10 for $\mathbf{PQ}^{=}$ is equivalent to a rule asserting that '$P[Fc \mid B] \geqslant P[b = c \,\&\, Fb \mid B]$'; and rule 10 could equally be replaced by the rule that '$P[(b = c \,\&\, Fb) \supset Fc \mid B] = 1$'. Similarly, rule 10 for $\mathbf{EQ}^{=}$ is equivalent to the rule 'if $B \rightarrow (b = c \,\&\, Fb)$, then $B \rightarrow Fc$', and also to the rule '$B \rightarrow (b = c \,\&\, Fb) \supset Fc$'. Definition 22 may strike you as a trivial way to introduce identity into the semantics. But recall that much of the point behind the way we've approached semantics in this paper is to *emulate* objectual semantics without having to work out the metaphysical details in advance – to get the form without assuming too much about the content. Definition 22 does just that.

One may sometimes find it useful to take some first-order theory to be part of the "background logic" for conditional probability functions in $\mathbf{PQ}^{=}$ (e.g. a version of applied ZF set theory with *urelements*, see Supppes (1972)). This is easily handled.

DEFINITION 23. For any set of sentences T in an $\mathcal{L}$-language L, define $\mathbf{PQ} + T(\mathbf{PQ}^{=} + T)$ as the set of all $P \in \mathbf{PQ}(\mathbf{PQ}^{=})$ on $\mathcal{L}$-languages containing the sentences in T such that for each sentence C in the language of P and all $B \in T$, $P[B \mid C] = 1$.

One may define $\mathbf{EQ} + T$ similarly (i.e. as the set of all $\rightarrow \in \mathbf{EQ}$ such that for each C in the language and for all $B \in T,\ C \rightarrow B$).

Notice that if D is a logical consequence of T, then it's easily shown that for each $P \in \mathbf{TQ} + T$ (or $\mathbf{PQ}^{=} + T$), $P[D \mid C] = 1$ (for all C in the language of T). And conversely, if for *every* $P \in \mathbf{PQ} + T$ (or $\mathbf{PQ}^{=} + T$), for all C in the language of P, $P[D \mid C] = 1$, then D

must be a logical consequence of T. These claims follow directly from Theorem 6 (and Definition 14).

There is a further way in which the semantic theories for $\mathbf{EQ}$ and $\mathbf{PQ}$ might be extended – a way that is of a very different kind. The semantics of $\mathbf{EQ}$ might be extended to provide a semantics for a language containing object-language nonmonotonic conditionals (e.g. object language subjunctive conditionals). And, similarly, $\mathbf{PQ}$ might be extended to provide a semantics for object-language probabilities (e.g. probabilities that represent the physical propensities of systems). The idea is that just as truth-value semantics explicates '&' and '$\neg$' in the object-language in terms of 'and', 'not', and 'is true' in the metalanguage, so some kinds of indicative and subjunctive object-language conditionals might be explicated in terms of metalinguistic conditionals in $\mathbf{EQ}$. For such a conditional '–>' in the object-language it may prove useful to extend $\mathbf{EQ}$ with a direct inference rule, something like: $((B \text{–>} A)\&B) \rightarrow A$. The tricky part will come in trying to specify the kinds of conditions C that should defeat the direct inference – i.e. the sentences C such that $(C\&((B \text{–>} A)\&B)) \not\rightarrow A$. For instance, if C is of form $(D\&((D\&B) \text{–>} \neg A))$, then, pretty clearly, C should defeat the direct inference.

Similar issues arise for semantic probability rules that govern object-language probabilities. In the present systems the only purely logical relationships that hold among sentences in PQ-models are logical entailment and a closely related notion. That is, the only conditional probability values on which all members P of $\mathbf{PQ}$ agree are these: $P[A \mid B] = 1$ when $B \vDash A$; $P[A \mid B] = 0$ when both $\vDash \neg A$ and $\vDash B$. Suppose we expand $\mathbf{PQ}$ to an object-language that contains a binary "function symbol" 'p' that represents object-language probabilities. For instance, 'p' might apply to pairs of predicates and map them to (object-language representations of) real numbers r in $[0, 1]$; so that '$p(Fx, Gx) = r$' says that the propensity for things with G-ness to exhibit F-ness is r. It would then be natural to supplement $\mathbf{PQ}$ with a direct inference rule: $P[Gc \mid Fc \,\&\, p(F, G) = r] = r$. Here again the trick will be to specify which sorts of sentences C can defeat the direct inference, and this will likely be quite difficult to work out satisfactorily.

NOTES

[1] My thanks to Chris Swoyer and to an anonymous referee for their very helpful comments.

[2] My only reason for employing the notation '$v[A]$' rather than sticking strictly to '$A \in v$' is that I want to maintain a rough analogy between the notation for truth-value

assignments and the usual notations for nonmonotonic conditionals and for conditional probabilities. That is, the nonmonotonic conditionals I will describe in a bit are semantic relations – they are sets of ordered pairs of sentences. And, although it is technically correct to write '$\langle B, A\rangle \in\to$', it is more usual to simply write '$B \to A$'. Similarly, for semantic conditional probabilities it is more usual to write '$P[A \mid B] = r$', but one could instead use the notation '$\langle A, B, r\rangle \in P$', where '$P$' represents a function from sentence pairs to real numbers between 0 and 1.

[3] A referee for *JPL* offered the following comment:

> What is the purpose of defining *TQ-logical truth* not only for sentences of $\mathcal{L}$, but also for sentences B that belong to an extension of $\mathcal{L}$ without belonging to $\mathcal{L}$ itself? I took it that the "root language" $\mathcal{L}$ is the language for which semantic notions are to be defined. What other purpose is there for singling out $\mathcal{L}$? If it is understood that B is a sentence of $\mathcal{L}$, and hence a sentence of every $\mathcal{L}$-language, then Definition 5 becomes
>
> B is a *TQ-logical truth iff* for all $v \in$ **TQ** defined on $\mathcal{L}$, $v[B]$.
>
> The definition of *consequence* and Definitions 9 and 14 can similarly be simplified. And Theorem 1 could state that the truth-value assignments $v \in$ **TQ** which are defined on $\mathcal{L}$ constitute the set of all objectual truth-value assignments on $\mathcal{L}$.

It seems to me that the referee's suggestion would work just fine. And the referee's point is right; it is the notions of *logical truth* and *logical consequence* for the "root language" that are of central interest. But, if I were to switch to the suggested alternative definitions, the reader might then wonder whether they could be generalized to apply to $\mathcal{L}$ and it's name extensions all at once, or whether there might be some reason for not doing so. Thus, having made the reader aware of these alternative renditions, I'll continue to employ the broader versions.

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