New Horn Rules for Probabilistic Consequence: Is O+ Enough?

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Abstract In our 2007 paper David and I studied consequence relations that correspond to conditional probability functions above thresholds, the *probabilistic consequence relations*. We showed that system O is a probabilistically sound system of Horn rules for the probabilistic consequence, and we conjectured that O might also provide a complete axiomatization of the set of finite premised Horn rules for probabilistic consequence relations. In a 2009 paper Paris and Simmonds provided a mathematically complex way to characterize all of the sound finite-premised Horn rules for the probabilistic consequence relations, and they established that the rules derivable from system O are insufficient. In this paper I provide a brief accounts of system O and the probabilistic consequence relations. I then show that O together with the probabilistically sound (Non-Horn) rule known as Negation Rationality implies an additional systematic collection of sound Horn rules for probabilistic consequence relations. I call O together with these new Horn rules 'O+'. Whether O+ is enough to capture all probabilistically sound finite premised Horn rules remains an open question.

Keywords Probabilistic consequence relation · Probability threshold · Horn rule · Nonmonotonic consequence

1 Introduction

In our 2007 paper, "The Quantitative/Qualitative Watershed for Rules of Uncertain Inference", David and I studied consequence relations that correspond to conditional probability functions above thresholds. That is, we studied the *probabilistic consequence relations* (hereafter the *ProbCR*s), defined as follows:

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S. O. Hansson (ed.), *David Makinson on Classical Methods for Non-Classical Problems*, Outstanding Contributions to Logic 3, DOI: 10.1007/978-94-007-7759-0_9,

Definition 1. Probabilistic Consequence Relations: Let p be any probability function defined on sentences of a language for sentential logic,[1] and let t be any real number such that $0 < t \leq 1$. The pair $< p, t >$ generates a probabilistic consequence relation $|\!\sim_{p,t}$ by the rule: $a|\!\sim_{p,t} x$ iff either $p(a) = 0$ or $p_a(x) \geq t$, where by definition $p_a(x) = p(x|a) = p(a \wedge x)/p(a)$ for $p(a) > 0$. The parameter t is called the *threshold* associated with p for the relation $|\!\sim_{p,t}$.

A leading idea of that paper was to show that a system of Horn rules called **O** lies at the cusp between probabilistically sound Horn rules and well-studied stronger qualitative systems that fail to be probabilistically sound. **O** consists of several familiar rules for nonmonotonic consequence relations together with some weakened versions of other familiar rules, all of them sound for the *ProbCR*s. Some other well-known rules, such as AND (if $a \,|\!\sim\, x$ and $a \,|\!\sim\, y$, then $a \,|\!\sim\, x \wedge y$) and OR (if $a \,|\!\sim\, x$ and $b \,|\!\sim\, x$, then $a \vee b \,|\!\sim\, x$), are unsound for the *ProbCR*s. However, **O** turns out to be strong enough that merely adding AND to **O** not only *jumps the divide* between probabilistically sound rules and stronger qualitative rules for uncertain inference, rather it takes the logic all the way over to the strong system called *preferential consequence relations*, which is usually characterized by the system of Horn rules called **P**. Nevertheless, strong as it is, **O** does not contain every rule that is sound for the *ProbCR*s. The well-know non-Horn rule NR (negation rationality: if $a \,|\!\sim\, x$, then either $a \wedge b \,|\!\sim\, x$ or $a \wedge \neg b \,|\!\sim\, x$) is also sound for *ProbCR*s. Adding NR to **O** results in the probabilistically sound system we call **Q**, also investigated in our 2007 paper.[2]

Our paper showed that no set of sound *finite-premised* Horn rules is *complete* for all *ProbCR*s.[3] For, we showed, there are *infinite-premised* Horn rules sound for *ProbCR*s that cannot be derived from any sound set of finite-premised Horn rules. This result left open the question of whether the sound finite-premised Horn rules we investigated (the system **O**) would suffice to derive all sound finite-premised Horn rules for *ProbCR*.

In the present paper, after summarizing the systems of sound rules for *ProbCR*, I'll extend some of the main ideas and results from our 2007 paper. These new results are based on some work David and I did after our 2007 paper was published. This new work was motivated by an exchange of email messages with Jeff Paris and Richard Simmonds, who contacted us after proving their important *completeness result* for a

[1] That is, p satisfies the usual classical probability axioms on sentence of a language for sentential logic: (1) $p(a) \geq 0$, (2) if $|\!-\, a$ (i.e. if a is a tautology), then $p(a) = 1$, (3) if $|\!-\, \neg(a \wedge b)$, then $p(a \vee b) = p(a) + p(b)$; and conditional probability is defined as $p(a \mid b) = p(a \wedge b)/p(b)$ whenever $p(b) > 0$. All of the other usual probabilistic rules follow from these axioms.

[2] The systems **O** and **Q**, and their probabilistic soundness, were first investigated in (Hawthorne 1996). The more recent paper, (Hawthorne and Makinson 2007), proves important new results about **O**, **Q**, and related systems.

[3] That is, any set of sound rules for *ProbCR*s that are in *Horn rule form* will be satisfied by some relations $|\!\sim$ on all pairs of sentences that are not in *ProbCR*s. A rule is in *Horn rule form* just when it is of form, "if $a_1|\!\sim x_1, \ldots, a_n|\!\sim x_n$, then $b|\!\sim y$" (with at most a finite number of premise conditions of form $a_1|\!\sim x_1, \ldots, a_n|\!\sim x_n$), and perhaps also containing *side conditions* about logical entailments among sentences.

characterization of the finite-premised Horn rules for *ProbCR*s. In their paper "**O** is Not Enough" (2009), Paris and Simmonds show how to capture all of the sound finite-premised Horn rules for *ProbCR*s, and they establish that the rules we investigated in our 2007 paper were *not enough*. Although Paris and Simmonds characterize a complete set of finite-premised Horn rules for *ProbCR*s, their characterization is fairly opaque. They provide an algorithm for generating all sound Horn rules (and prove that it does so). But the algorithm for generating the rules is complex enough that it's not at all easy to tell what the rules it generates will look like in advance of just cranking them out one at a time—and there are an infinite number of them to crank out.

Motivated by the Paris and Simmonds result, David and I have discovered an explicit infinite sequence of additional sound Horn rules. But it isn't clear whether these new rules suffice to derive all finite-premised Horn rules—i.e. to derive all of those rules generated by the Paris-Simmonds procedure. The present paper is devoted to specifying these additional rules and raising unsolved questions about *complete rules* for *ProbCR*s.

2 Probabilistic Consequence Relations and the O and Q Rules: A Quick Overview of Earlier Results

The family **O** of rules for consequence relations is defined as follows.

Definition 2. The family of rules **O**:

$a \mid\sim a$	(REFLEX: reflexivity)
If $a \mid\sim x$ and $x \mid- y$, then $a \mid\sim y$	(RW: right weakening)
If $a \mid\sim x$ and $a \mid- b$ and $b \mid- a$, then $b \mid\sim x$	(LCE: left classical equivalence)
If $a \mid\sim x \wedge y$, then $a \wedge x \mid\sim y$	(VCM: very cautious monotony)
If $a \wedge b \mid\sim x$ and $a \wedge \neg b \mid\sim x$, then $a \mid\sim x$	(WOR: weak OR)
If $a \mid\sim x$ and $a \wedge \neg y \mid\sim y$, then $a \mid\sim x \wedge y$	(WAND: weak AND).

These rules are sound for the *ProbCR*s (see Hawthorne and Makinson 2007; Hawthorne 1996). Furthermore, given the other rules, the WOR could be replaced by the following rule.

If $\mid- \neg(a \wedge)$ and $a \mid\sim x$ and $b \mid\sim x$, then $(a \vee b) \mid\sim x$ (XOR: exclusive OR).

That is, given the other rules, one can derive XOR from WOR, and *vice versa*. This is especially interesting because it turns out that the additional sound Horn rules we've discovered for *ProbCR* are extended versions of the XOR rule. I'll get to those in the next section.

The rules in **O** are quite similar to the rules in the family **P**, which is a sound and complete family of rules for the *preferential consequence relations*. These consequence relations are defined semantically in terms of *stoppered* (a.k.a. *smooth*) *preferential models* (see Krauss et al. 1990; Makinson 1989, 1994).

Definition 3. The family of rules **P**:
REFLEX, LCE, RW together with the following rules:

If $a|\sim x$ and $a|\sim y$, then $a \wedge x|\sim y$ (CM: cautious monotony)
If $a|\sim x$ and $b|\sim x$, then $a \vee b|\sim x$ (OR : disjunction in the premises)
If $a|\sim x$ and $a|\sim y$, then $a|\sim x \wedge y$ (AND : conjunction in conclusion).

Notice that each of the last three rules for **P** is a stronger version of the corresponding rules for **O**. However, the OR rule looks rather more like the rule XOR than like WOR. Furthermore, some versions of **P** use the following rule as a basic rule in place of AND.

If $a|\sim x$ and $a \wedge x|\sim y$, then $a|\sim y$ (CT: cumulative transitivity (a.k.a. CUT)).

Given the other rules, CT and AND are derivable from each other.

It turns out that the usual rules for family **P** are stronger than necessary. That is, consider the following family of rules.

Definition 4. The family of rules $\mathbf{P}^*$ consists of the rules of **O** with WAND replaced by AND.

David and I showed that all of the rules of **P** are derivable from those of $\mathbf{P}^*$, and *vice versa* (also see Hawthorne 1996). Thus, the rule AND is the watershed rule that takes one from the *ProbCR* to the *preferential consequence relations*. Furthermore, AND is clearly not sound for *ProbCR*, because it's often the case that $p(x \wedge y|a) < p(x|a)$. So, when either $p(x|a)$ or $p(y|a)$ is very close to the threshold t for a relation $|\sim_{p,t}$, it can be the case that $p(x \wedge y|a) < t$.

Not all sound rules for *ProbCR*s are Horn rules. Indeed, as already mentioned in the Introduction, the well-known rule called *negation rationality* is sound for *ProbCR*s.

If $a|\sim x$, then either $a \wedge b|\sim x$ or $a \wedge \neg b|\sim x$ (NR).

Definition 5. The family of rules $\mathbf{Q} : \mathbf{Q} = \mathbf{O} \cup \{\text{NR}\}$.

All rules of **Q** are sound for *ProbCR*s.

Although David and I show that **Q** is sound, what we hadn't realized at that time is the surprising result that the non-Horn rule NR permits the derivation in system **Q** of infinitely many additional Horn rules that are not derivable from **O** alone.

Before moving on to the new results, one additional point is worth making clear. It is easy to specify an infinite-premised Horn rule that is sound for *ProbCR*s. Just consider any infinite set of distinct sentences $\{x_1, x_2, \ldots, x_n, \ldots\}$ such that for any pair of them, $a|\!-\neg(x_i \wedge x_j)$. Then it's not possible to have a probability function p and threshold $t > 0$ such that for each x_i, $p(x_i|a) \geq t$. For, suppose there is such a p and $t > 0$. Then there is an integer $n > 1$ such that $t > 1/\text{n}$; so that $n \times t > 1$. Now notice that $1 = p(x_1 \vee \neg x_1|a) = p(x_1|a) + p(\neg x_1|a) = p(x_1|a) + p(\neg x_1 \wedge (x_2 \vee \neg x_2)|a) = p(x_1 \mid a) + p(\neg x_1 \wedge x_2|a) + p(\neg x_1 \wedge \neg x_2|a) = p(x_1|a) + p(x_2|a) + p(\neg x_1 \wedge \neg x_2|a) = \ldots = p(x_1 \mid a) + p(x_2|a) + \ldots + p(x_n|a) + p(\neg x_1 \wedge \neg x_2 \wedge \ldots \wedge \neg x_n|a) \geq n \times t > 1$. Contradiction! Thus, the only way to have a *ProbCR* for which $a|\!\sim x_i$ for an infinite set $\{x_1, x_2, \ldots, x_n, \ldots\}$ where $a \vdash \neg(x_i \wedge x_j)$ for each pair of them is to have $p(a) = 0$; that's the degenerate case where $a|\!\sim_{p,t}\, y$ for all y (including $\bot$). Thus, the following rule is sound for *ProbCR*:

If for each x_k in $\{x_1, x_2, \ldots, x_n, \ldots\}$, $a|\!\sim x_k$, and for each pair, $a \vdash \neg(x_i \wedge x_j)$, then $a|\!\sim \bot$ (INF).

A number of such infinitary Horn rules are sound for *ProbCR*s. The "Archimedean rule" we presented in our paper is a more complex variation on the same idea. However, such rules are not really that "troubling" for the project of characterizing *ProbCR*s because for any specific threshold *t*, a finite Horn rule will subsume the infinitary rule. That is, let's define *ProbCR*(q) as the set of all *ProbCR*s for which the threshold $t > q$.

Definition 6. Probabilistic Consequence Relations for thresholds above q. Let *ProbCR*(q) be the set of all *ProbCR*s, $|\!\sim_{p,t}$, such that the threshold $t > q > 0$.

In our paper David and I explored sound threshold-sensitive rules for various threshold levels q. However, these rules turn out to be rather complex, so I'll not discuss them in any detail here. The point is that for any given threshold $t \geq \text{q} > 1/n$ for integer $n > 1$, the following finite-premised Horn rule will be sound, and will subsume the INF rule.

If $a|\!\sim x_1, a|\!\sim x_2, \ldots, a|\!\sim x_n$, and for each pair, $a \vdash \neg(x_i \wedge x_j)$, then $a|\!\sim \bot$ (PLAUS(n)).

Given the fact that all known infinite-premised Horn rules that are sound for *ProbCR*s are subsumable by finite-premised rules that are sound for greater than zero bounds on thresholds, perhaps only finite-premised Horn rules should be of any real interest for *ProbCR*s.

In any case, in our 2007 paper David and I conjectured that **O** might suffice for generating all sound finite-premised Horn rules for *ProbCR*s. Within the year after our paper was published, Paris and Simmonds proved otherwise. Although their result did not came out until 2009, they contacted us about their result in February of 2008, and got us thinking.

3 Why O isn't Enough

The Paris and Simmonds (2009) paper is brilliant, and mathematically extremely sophisticated. They provide an algorithm for generating the complete set of sound finite-premised Horn rules for *ProbCR*s, and prove that it does so. But it's difficult to see what these rules look like without simply generating them one at a time. However, their algorithm establishes that there must be an infinite set of sound independent rules (not derivable from any finite subset of the rules). Indeed, for each positive natural number *n*, there must be such a rule consisting of at least *n* premises, a rule of the form

$$\text{If } a_1 |\!\sim x_1, a_2 |\!\sim x_2, \ldots, a_n |\!\sim x_n, \text{ then } b |\!\sim y$$

together with side conditions about logical entailments among the various sentences involved.

Paris and Simmonds (2009) show that all rules of **O** are generated via the first few iterations of their algorithm. All of the additional examples *not* derivable from **O** that they have explicitly generated are very similar to the following example:

$$\text{If } (a \wedge \neg b) \vee (b \wedge \neg a) |\!\sim x, a |\!\sim x, b |\!\sim x, \text{ then } a \vee b |\!\sim x \; (\text{PS}).$$

Let's call this the PS rule for "Paris-Simmonds". Some of their examples have more premises, but all that we are aware of are analogous in a way I'll explain in a moment.

Upon seeing such examples, it occurred to me and David that these examples are somewhat like the XOR rule, which is derivable from **O**. To see the pattern, consider:

$$\text{If } |\!- \neg(a \wedge b) \text{ and } a |\!\sim x \text{ and } b |\!\sim x, \text{ then } (a \vee b) |\!\sim x \quad (\text{XOR}).$$

Now, put the PS rule into the following form:

$$\text{If } (a \wedge \neg b) \vee (\neg a \wedge b) |\!\sim x, (a \wedge b) \vee (a \wedge \neg b) |\!\sim x, (a \wedge b) \vee (\neg a \wedge b) |\!\sim x, \\ \text{then } (a \wedge b) \vee (a \wedge \neg b) \vee (\neg a \wedge b) |\!\sim x (\text{PS}).$$

Here, the antecedent of each premise is logically equivalent to the antecedent in the original version of the PS rule, above. But this version makes it clear that the antecedent of the conclusion is simply the disjunction of all disjuncts from antecedents of the premises. Furthermore, in this case each of the disjuncts is mutually inconsistent with each of the other disjuncts. This suggests the following rule, which is analogous to XOR but not derivable from it:

$$\text{If } |\!- \neg(a \wedge b), |\!- \neg(a \wedge c), |\!- \neg(b \wedge c), \text{ and, } a \vee b |\!\sim x, a \vee c |\!\sim x, b \vee c |\!\sim x, \\ \text{then } (a \vee b \vee c) |\!\sim x \; (\text{XOR}[3, 2]).$$

I call this rule XOR [3, 2] because it goes from the support of x by pairs of exclusive disjuncts to the support of x by all three disjuncts. By this labeling scheme the usual XOR rule is XOR [2, 1].

Is this rule sound for *ProbCR*s? It turns out that it is, and the easiest way to prove that is to derive it from the rules of **Q**. In particular, the non-Horn rule *negation rationality*, NR, provides just the boost **O** needs to permit a derivation of this new rule. And since all of the **Q** rules are sound for *ProbCR*, whatever rules we derive from **Q** will also be sound.

To see how NR helps with the derivation of XOR [3, 2], let's first derive an alternative version of NR:

$$\text{If} \mid\!- \neg(a \wedge b) \text{ and } a \vee b \mid\!\sim x, \text{ then either } a \mid\!\sim x \text{ or } b \mid\!\sim x \qquad (\text{XNR}: \text{ eXclusiveNR}).$$

Here's the derivation of XNR from NR:

Suppose $\mid\!- \neg(a \wedge b)$ and $a \vee b \mid\!\sim x$. Then, from NR, either $(a \vee b) \wedge a \mid\!\sim x$ or $(a \vee b) \wedge \neg a \mid\!\sim x$. Thus, since $b \mid\!- \neg a$, we have, either $a \mid\!\sim x$ or $b \mid\!\sim x$, via LCE applied to each antecedent.

The implication goes the other way, from XNR to NR, as well. Here is that direction from O ∪ {XNR} (actually we only need LCE, as in the previous proof):

Suppose $a \mid\!\sim x$. From LCE we have $(a \wedge b) \vee (a \wedge \neg b) \mid\!\sim x$. Then, since $\mid\!- \neg((a \wedge b) \wedge (a \wedge \neg b))$, from (XNR) we get, either $(a \wedge b) \mid\!\sim x$ or $(a \wedge \neg b) \mid\!\sim x$.

Now, I brought up XNR is as a means of proving that XOR [3, 2] is sound for *ProbCR*, by deriving XOR[3, 2] from **Q** via XNR. That derivation is pretty straightforward.

Observation 1: XOR[3, 2] is sound for *ProbCR*.

Proof: Suppose $\mid\!- \neg(a \wedge b)$, $\mid\!- \neg(a \wedge c)$, $\mid\!- \neg(b \wedge c)$, and $a \vee b \mid\!\sim x$, $a \vee c \mid\!\sim x$, $b \vee c \mid\!\sim x$. From $a \vee b \mid\!\sim x$ we get (by XNR) that either (i) $a \mid\!\sim x$ or (ii) $b \mid\!\sim x$. In case (i) we apply XOR together with $b \vee c \mid\!\sim x$ to get $a \vee b \vee c \mid\!\sim x$ (since it follows from the side conditions that $\mid\!- \neg(a \wedge (b \vee c))$). Similarly, in case (ii) we apply XOR together with $a \vee c \mid\!\sim x$ to get $a \vee b \vee c \mid\!\sim x$ (since it follows from the side conditions that $\mid\!- \neg(b \wedge (a \vee c))$). Thus, XOR[3, 2] follows from sound rules for *ProbCR*.

The structure of the XOR[3,2] rule and the XOR[2,1] rule, (a.k.a. the XOR rule) suggests a host of much more general rules of the following sort.

For each pair of integers n, m such that $n > m \geq 1$, define the rule XOR (n, m) as follows:

XOR (n, m): Consider any list of n pairwise inconsistent sentences. Suppose that for each sentence ***e*** that consists of a disjunction (in the order provided by the list, just to be concrete about it) of exactly m of them we have $e \mid\!\sim x$. Then, for the sentence d that consists of the disjunction of all n of them (in the order provided by the list) it follows that $d \mid\!\sim x$.

All of the XOR(n, m) rules (for $n > m \geq 1$) turn out to be sound for *ProbCR*. However, these rules are not independent. It turns out that from the set of sound Horn rules **O** together with only the new rules of form XOR($n+1$, n) for each $n \geq 2$ (since XOR(2, 1) is already part of **O**) we can derive all of the other XOR(n, m) rules.

To establish these claims we'll proceed as follows. I'll first define the set of Horn rules **O+**, which consists of **O** together with the XOR($n + 1$, n) rules, for each $n \geq 2$. We then show that all of the new XOR($n + 1$, n) rules follow from **Q**. That will establish the soundness of **O+** for *ProbCR*s. Then we show that the remaining XOR(n, m) rules, for each $n > m \geq 1$, are derivable in **O+**.

4 The Soundness of the Horn Rule System O+ and the Derivation of the XOR(n, m) Rules

Definition 7. The family of rules **O+** : **O+** = **O** $\cup$ {xor($n + 1$, n) rules for each $n \geq 2$}.

Observation 2: The family of rules **O+** is sound for *ProbCR*s.

We establish the soundness of **O+** by showing that each XOR($n + 1$, n) rules for each $n \geq 1$ follows from the sound set of rules **Q**. The proof is by induction on n.

Proof: basis: XOR(2,1) is just XOR, which has already been established as sound for ProbCRs.

Induction hypothesis: Now, suppose that for all k such that $1 \leq k \leq n - 1$, the rules XOR($k + 1$, k) hold. (We show that XOR($n + 1$, n) must also hold.)

Induction step: Let the members of the list $< a_1, \ldots, a_{n-1}, a_n, a_{n+1} >$ consist of pairwise inconsistent sentences, and suppose that for each disjunction (in order) of any n of them, e, we have $e \mid\!\sim x$. (We want to show that for the disjunction of all of them, $a_1 \vee \ldots \vee a_{n-1} \vee a_n \vee a_{n+1} \mid\!\sim x$.)

Notice that $a_1 \vee \ldots \vee a_{n-1} \vee a_n \mid\!\sim x$ and $a_1 \vee \ldots \vee a_{n-1} \vee a_{n+1} \mid\!\sim x$. Applying XNR to $a_1 \vee \ldots \vee a_{n-1} \vee a_{n+1} \mid\!\sim x$ yields that either $a_1 \vee \ldots \vee a_{n-1} \mid\!\sim x$ or $a_{n+1} \mid\!\sim x$.

(i) Suppose $a_{n+1} \mid\!\sim x$. Then, since $\vdash \neg(a_{n+1} \wedge (a_1 \vee \ldots \vee a_{n-1} \vee a_n))$, putting this with $(a_1 \vee \ldots \vee a_{n-1} \vee a_n) \mid\!\sim x$ via XOR(2, 1) yields $a_1 \vee \ldots \vee a_{n-1} \vee a_n \vee a_{n+1} \mid\!\sim x$, and we are done.

(ii) Alternatively, if $a_{n+1} \mid\!\sim x$, then $a_1 \vee \ldots \vee a_{n-1} \mid\!\sim x$. Now, consider the sequence of n sentences $S = <a_1, \ldots, a_{n-1}, (a_n \vee a_{n+1})>$. The members of S are pairwise inconsistent (since $\vdash \neg(a_j \wedge (a_n \vee a_{n+1}))$ holds for each j such that $1 \leq j \leq n - 1$). For each a_j with $1 \leq j \leq n - 1$ we already had that $a_1 \vee \ldots \vee a_{j-1} \vee a_{j+1} \vee \ldots \vee a_{n-1} \vee (a_n \vee a_{n+1}) \mid\!\sim x$ (this is a supposition of the present induction step), and we also have $a_1 \vee \cdots \vee a_{n-1} \mid\!\sim x$. So, $S = <a_1, a_2, \ldots, a_{n-1}, (a_n \vee a_{n+1})>$ is a list of n pairwise inconsistent sentences , where for each ordered sequence e of $n - 1$ of them, we have $e \mid\!\sim x$. Thus, by the induction hypothesis, for the disjunction of all of them, $a_1 \vee \ldots \vee a_{n-1} \vee (a_n \vee a_{n+1})|\sim x$. Then $a_1 \vee \ldots \vee a_{n-1} \vee a_n \vee a_{n+1}|\sim x$.

Now let's establish that all of the XOR(n, m) rules (for $n > m \geq 1$) are derivable in **O+**.

Observation 3: From **O+** = **O** $\cup$ {XOR($n+1, n$) rules for each n ≥ 2} it follows that all XOR(n, m) rules hold for all $n > m \geq 1$.

Proof: Let m be any natural number such that $m \geq 1$.

(1) Basis: $n = \mathrm{m} + 1$: Let S be any sequences of $n = m + 1$ pairwise inconsistent sentences such that for each disjunction e of m members of S (in the order specified by S) we have $e|\sim x$. Then by XOR($m+1, m$) we have, for the disjunction d of all members of S (in the order specified by S), $d|\sim x$.

(k) Induction hypothesis: $n = m+k$: Suppose that for any sequence S of $n = m+k$ pairwise inconsistent sentences, if for each disjunction e of m members of S (in the order specified by S) we have $e\ |\sim x$, then also we have, for the disjunction d of all members of S (in the order specified by S), $d|\sim x$.

(k+1) Induction step: $n = m + k + 1$: Let S be any sequences of $n = m + k + 1$ pairwise inconsistent sentences such that for each disjunction e of m members of S (in the order specified by S) we have $e|\sim x$.
Let S^* be any subsequence of S consisting of $m + k$ members of S. S^* is a sequence of $m + k$ inconsistent sentences, where, for each disjunction e of m members of S^* (in the order specified by S) we have $e\ |\sim x$. So for the disjunction d of all $m + k$ members of S^* (in the order specified by S) we have $d|\sim x$.
So, by the induction hypothesis, for each disjunction d of $m + k$ members of S (in the order specified by S) we have $d|\sim x$. Then by XOR($m+k+1, m+k$) we have, for the disjunction d^+ of all members of S (in the order specified by S), $d^+|\sim x$.

Thus, all of the XOR(n, m) rules (for all $n > m \geq 1$) are sound for the *ProbCR*s.

5 Is O+ Enough? If Not, Then How About Q?

Notice that the Family **O+** provides an infinite list of new Horn rules sound for *ProbCR*s. Paris and Simmonds established that **O** isn't enough by itself, and that only an infinite list of rules can provide a set of axioms that are sufficiently complete to permit the derivation of every finite-premised Horn rules that is sound for *ProbCR*s. Furthermore, **O+** provides derivations for all of the specific rules we know of that have been explicitly calculated via the Paris-Simmonds algorithm. So perhaps **O+** is enough. Is it? That's the central outstanding issue for *ProbCR*s at present.

If the answer to this question turns out to be negative, then the further question remains: Is **Q** enough? For, **O+** is derivable from the *ProbCR* sound rules that make up **Q**, even though the extra power of **Q** comes from the non-Horn rule NR. So, whatever the complete set of finite-premised Horn rules may be, perhaps **Q** suffices to derive them all.

References

Hawthorne, J. (1996). On the logic on non-monotonic conditionals and conditional probabilities. *Journal of Philosophical Logic*, *25*, 185–218.

Hawthorne, J., & Makinson, D. (2007). The qualitative/quantitative watershed for rules of uncertain inference. *Studia Logica*, *86*, 247–297.

Krauss, S., Lehmann, D., & Magidor, M. (1990). Nonmonotonic reasoning preferential models and cumulative logics. *Artificial Intelligence*, *44*, 167–207.

Makinson, D., (1989). General Theory of Cumulative Inference. In M. Reinfrank, J. de Kleer, M. L. Ginsberg, & E. Sandewall (Eds.) *Non-Monotonic Reasoning—Proceedings of the 2nd International Workshop 1988* (pp. 1–18). Berlin: Springer.

Makinson, D. (1994). General Patterns in Nonmonotonic Reasoning. In Dov M. Gabbay, C. J. Hogger, & J. A. Robinson (Eds.), *Handbook of Logic in Artificial Intelligence and Logic Programming, Non-Monotonic and Uncertainty Reasoning* (Vol. 3, pp. 35–110). Oxford: Oxford University Press.

Paris, J., & Simmonds, R. (2009). O is not enough. *Review of Symbolic Logic*, *2*, 298–309.