An Even Better Solution to the Paradox of the Ravens

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Think of confirmation in the context of the Ravens Paradox this way. The likelihood ratio measure of incremental confirmation gives us, for an observed Black Raven and for an observed non-Black non-Raven, respectively, the following "full" likelihood ratios:[1]

$$\frac{P[Bc\cdot Rc \mid H\cdot K]}{P[Bc\cdot Rc \mid {\sim}H\cdot K]} = \frac{P[Bc \mid Rc\cdot H\cdot K]}{P[Bc \mid Rc\cdot{\sim}H\cdot K]} \times \frac{P[Rc \mid H\cdot K]}{P[Rc \mid {\sim}H\cdot K]}$$

$$= (1/ P[Bc \mid Rc\cdot{\sim}H\cdot K]) \times (P[Rc \mid H\cdot K]/P[Rc \mid {\sim}H\cdot K]) \quad \text{and}$$

$$\frac{P[{\sim}Bc\cdot{\sim}Rc \mid H\cdot K]}{P[{\sim}Bc\cdot{\sim}Rc \mid {\sim}H\cdot K]} = \frac{P[{\sim}Rc \mid {\sim}Bc\cdot H\cdot K]}{P[{\sim}Rc \mid {\sim}Bc\cdot{\sim}H\cdot K]} \times \frac{P[{\sim}Bc \mid H\cdot K]}{P[{\sim}Bc \mid {\sim}H\cdot K]}$$

$$= (1/ P[{\sim}Rc \mid {\sim}Bc\cdot{\sim}H\cdot K]) \times (P[{\sim}Bc \mid H\cdot K]/P[{\sim}Bc \mid {\sim}H\cdot K]).$$

Each "full" likelihood ratio decomposes into two parts, a "standard" likelihood ratio, and a second likelihood ratio of a "non-standard" sort, involving what may be considered the experimental or observation conditions.

In the case where a Black Raven is in evidence, the *standard likelihood ratio* compares the *likelihood* that a given Raven should turn out to be Black when H is true to its *likelihood* when H is false. The *non-standard likelihood ratio* compares the *likelihood* that a Raven should come to be in evidence at all when H is true, given background knowledge K, to its likelihood when H is false. The *full likelihood ratio* is just the product of these two parts.

Similarly, when a non-Black non-Raven is in evidence, the *standard likelihood ratio* compares the *likelihood* that a given non-Black object should turn out to be a non-Raven when H is true to its *likelihood* when H is false. And the *non-standard likelihood ratio* compares the *likelihood* that a non-Black object should come to be in evidence at all when H is true, given background knowledge K, to its likelihood when H is false. And again, the *full likelihood ratio* is just the product of these two parts.

Although Bayesians sometimes ignore this *non-standard likelihood ratio*, consisting of the experimental or observation conditions, Bayes' theorem actually requires this ratio. To see this, just have a look at the ratio form (i.e. odds form) of Bayes's theorem:

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[1] 'H' is our abbreviation for the Ravens hypothesis, "All Ravens are Black": $\forall x(Rx \supset Bx)$. K is relevant background knowledge. The meanings of the remaining notation should be obvious.

$$\frac{P[H \mid Bc \cdot Rc \cdot K]}{P[\sim H \mid Bc \cdot Rc \cdot K]} = \frac{P[Bc \mid Rc \cdot H \cdot K]}{P[Bc \mid Rc \cdot \sim H \cdot K]} \times \frac{P[H \mid Rc \cdot K]}{P[\sim H \mid Rc \cdot K]}$$

$$= \frac{P[Bc \mid Rc \cdot H \cdot K]}{P[Bc \mid Rc \cdot \sim H \cdot K]} \times \frac{P[Rc \mid H \cdot K]}{P[Rc \mid \sim H \cdot K]} \times \frac{P[H \mid K]}{P[\sim H \mid K]}$$

One might try to hide the *non-standard* ratio factor by burying Rc in K. But then K is just some background knowledge K* that has been updated with Rc; so in effect P[H | K] = P[H | Rc·K*]; so updating on Rc has occurred without explicit acknowledgement. Or one might just *assume* that Rc on its own (i.e., without Bc or ~Bc) makes no difference, that P[H | Rc·K] = P[H | K]. This *assumption* may provide an easy resolution of the Ravens paradox, but our approach will be more general. Taking P[H | Rc·K] to equal P[H | K] is just a special case of our resolution.

At a glance you can see that for the incremental confirmation of H due to a Black Raven, (Bc·Rc), to beat out the incremental confirmation due to a non-Black non-Raven, (~Bc·~Rc), the following two conditions would suffice:

(1) the *standard* likelihood ratio for a given Raven to turn out Black,
$$P[Bc \mid Rc \cdot H \cdot K] / P[Bc \mid Rc \cdot \sim H \cdot K] = 1 / P[Bc \mid Rc \cdot H \cdot K],$$
is larger than
the *standard* likelihood ratio for a given non-lack object to turn out to be a non-Raven,
$$P[\sim Rc \mid \sim Bc \cdot H \cdot K] / P[\sim Rc \mid \sim Bc \cdot \sim H \cdot K] = 1 / P[\sim Rc \mid \sim Bc \cdot H \cdot K] ;$$

(2) the incremental confirmation due to a non-Black object on its own,
$$P[\sim Bc \mid H \cdot K] / P[\sim Bc \mid \sim H \cdot K],$$
is not so much larger than
the incremental confirmation due to a Raven on its own,
$$P[Rc \mid H \cdot K] / P[Rc \mid \sim H \cdot K]$$
that it swamps the influence of the *standard* likelihood ratios in clause (1).

Indeed, conditions (1) and (2) as we've just stated them are stronger than required. All that's really needed is for the *standard* likelihood ratios in clause (1) to combine with the *non-standard* likelihood ratios of clause (2) in such as way as to make the *full* likelihood ratio due to a Black Raven larger than the *full* likelihood ratio due to a non-Black non-Raven. That is, we'll have

$$\frac{P[Bc \cdot Rc \mid H \cdot K]}{P[Bc \cdot Rc \mid \sim H \cdot K]} > \frac{P[\sim Bc \cdot \sim Rc \mid H \cdot K]}{P[\sim Bc \cdot \sim Rc \mid \sim H \cdot K]} \quad \textit{just in case we have}$$

$$\frac{P[Bc \mid Rc \cdot H \cdot K]}{P[Bc \mid Rc \cdot \sim H \cdot K]} \times \frac{P[Rc \mid H \cdot K]}{P[Rc \mid \sim H \cdot K]} > \frac{P[\sim Rc \mid \sim Bc \cdot H \cdot K]}{P[\sim Rc \mid \sim Bc \cdot \sim H \cdot K]} \times \frac{P[\sim Bc \mid H \cdot K]}{P[\sim Bc \mid \sim H \cdot K]} .$$

Our resolution of the Ravens paradox will draw on a way to characterize the sizes of the *standard* likelihood ratios, together with an assessment of how this bears on the permitted sizes of the *non-standard* ratios. This approach leads to an exceptionally general resolution of the Ravens issue. It turns out to provide quite plausible *necessary and sufficient conditions* for a Black Raven to confer more incremental support on "All ravens are black" than does a non-Black non-Raven.

To be perfectly rigorous about all this let's first set down some very weak conditions that suffice to make all the conditional probabilities we'll be dealing with well-defined, so that no divisions by zero occur in the definitions of any conditional probabilities on which we'll be drawing. These conditions also suffice to avoid completely trivial confirmational contexts for the Ravens situation – e.g. they suffice to make the prior probability of "All ravens are Black" greater than 0 and less than 1 relative to background knowledge K; and they guarantee the *possibility* that the evidence my turn up a Black Raven, or a non-Black Raven, or a non-Black non-Raven.

Non-Triviality Assumptions: $P[\sim Bc \cdot Rc \mid K] > 0$, and

$$P[Bc \cdot Rc \mid K] > P[(Bc \cdot Rc) \cdot \sim H \mid K] > 0, \text{ and } P[\sim Bc \cdot \sim Rc \mid K] > P[(\sim Bc \cdot \sim Rc) \cdot \sim H \mid K] > 0.$$

The conditional probability $P[Bc \mid Rc \cdot \sim H \cdot K]$ (that object c is Black given that it's a Raven and that the Ravens hypothesis H is false) will play an important role in our analysis. So let's label it by letting 'p' represent its value: $p = P[Bc \mid Rc \cdot \sim H \cdot K]$. The Non-Triviality Assumptions imply that $0 < p < 1$.[2]

Our analysis will draw on the *relative sizes* of the probabilities of an object turning out to be non-Black as compared to it turning out to be a Raven, when the ravens hypothesis H (together with background K) is *true*. We label the numerical value of this ratio with the letter 'r': $r = P[\sim Bc \mid H \cdot K] / P[Rc \mid H \cdot K]$. Presumably the value of r should be quite large. However, our analysis will not presuppose that this is so. The Non-Triviality Assumptions guarantee that the denominator $P[Rc \mid H \cdot K] > 0$ and that the numerator $P[\sim Bc \mid H \cdot K] > 0$. Thus, r is "well-defined" in the sense that the denominator of its definition cannot take the value 0; and $r > 0$. That's all we'll presuppose about r for now. In particular, although it may be reasonable to suppose that r is very much larger than 1 (perhaps a million to 1, or larger), we won't presuppose that here. Rather, we'll explicitly state any such supposition when it becomes relevant to our analysis.

Our analysis will also draw on the *relative sizes* of the probabilities of an object turning out to be non-Black as compared to it turning out to be a Raven, when the ravens hypothesis H (together with background K) is *false*. We'll label the numerical value of this ratio with the letter 'q': $q = P[\sim Bc \mid \sim H \cdot K] / P[Rc \mid \sim H \cdot K]$. The Non-Triviality Assumptions guarantee that the denominator $P[Rc \mid \sim H \cdot K] > 0$ and that the numerator $P[\sim Bc \mid \sim H \cdot K] > 0$. Thus, q is also "well-defined" in the sense that the denominator of its definition cannot take the value 0. We'll presuppose nothing else about q. Although it may be reasonable to take q to be a lot larger than 1, we won't presuppose that, but will explicitly state this supposition in cases where it becomes

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[2] Because $P[(Bc \cdot Rc) \cdot \sim H \mid K] > 0$ and (since $\sim Bc \cdot Rc \models \sim H$) $P[\sim Bc \cdot Rc \cdot \sim H \mid K] = P[\sim Bc \cdot Rc \mid K] > 0$, so $P[Rc \cdot \sim H \mid K] = P[Bc \cdot Rc \cdot \sim H \mid K] + P[\sim Bc \cdot Rc \cdot \sim H \mid K] > P[Bc \cdot Rc \cdot \sim H \mid K] > 0$, so $1 > P[(Bc \cdot Rc) \cdot \sim H \mid K]/P[Rc \cdot \sim H \mid K] > 0$, so $1 > P[Bc \mid Rc \cdot \sim H \cdot K] > 0$.

relevant to our analysis. One additional point about q: it follows from the Non-Triviality Assumptions that $q > 1-p > 0$. (This and other such claims are proved in the Appendix.)

The following theorem provides all of the essential ingredients of our resolution of the Ravens issue. (Non-Triviality implies that all conditional probabilities we draw on here and in the proof of this theorem are well-defined and greater than zero. All results are proved in the Appendix.)

Ravens Theorem. Non-Trivality together with the definitions of factors p, q, and r suffice for:

$$r = P[\sim Bc \mid H\cdot K]/P[Rc \mid H\cdot K] > 0; \quad q = P[\sim Bc \mid \sim H\cdot K]/P[Rc \mid \sim H\cdot K] > 0;$$

$$1 > p = P[Bc \mid Rc\cdot\sim H\cdot K] > 0; \quad q > (1-p) > 0;$$

$$\frac{P[Bc \mid Rc\cdot H\cdot K]}{P[Bc \mid Rc\cdot\sim H\cdot K]} = 1/p > 1 \; ; \qquad \frac{P[\sim Rc \mid \sim Bc\cdot H\cdot K]}{P[\sim Rc \mid \sim Bc\cdot\sim H\cdot K]} = 1/(1-(1-p)/q) > 1 \; ;$$

$$\frac{P[Bc \mid Rc\cdot H\cdot K]}{P[Bc \mid Rc\cdot\sim H\cdot K]} \Big/ \frac{P[\sim Rc \mid \sim Bc\cdot H\cdot K]}{P[\sim Rc \mid \sim Bc\cdot\sim H\cdot K]} = \frac{1-(1-p)/q}{p} = \frac{1-(1-p)/q}{1-(1-p)} \; ;$$

$$\begin{aligned} &> 1 \text{ when } q > 1, \\ &= 1 \text{ when } q = 1, \\ &< 1 \text{ when } q < 1 \;\; \{\text{note: } q > (1-p)\}; \end{aligned}$$

$$\frac{P[Rc \mid H\cdot K]}{P[Rc \mid \sim H\cdot K]} \Big/ \frac{P[\sim Bc \mid H\cdot K]}{P[\sim Bc \mid \sim H\cdot K]} = q/r \; ;$$

$$\frac{P[Bc\cdot Rc \mid H\cdot K]}{P[Bc\cdot Rc \mid \sim H\cdot K]} \Big/ \frac{P[\sim Bc\cdot\sim Rc \mid H\cdot K]}{P[\sim Bc\cdot\sim Rc \mid \sim H\cdot K]}$$

$$= \frac{1-(1-p)/q}{p} \times \frac{P[Rc \mid H\cdot K]}{P[Rc \mid \sim H\cdot K]} \Big/ \frac{P[\sim Bc \mid H\cdot K]}{P[\sim Bc \mid \sim H\cdot K]}$$

$$= [q-(1-p)]/(r\times p) \; ;$$

$$\frac{P[Bc\cdot Rc \mid H\cdot K]}{P[Bc\cdot Rc \mid \sim H\cdot K]} > \frac{P[\sim Bc\cdot\sim Rc \mid H\cdot K]}{P[\sim Bc\cdot\sim Rc \mid \sim H\cdot K]} \quad \textit{if and only if}$$

$$\frac{P[Rc \mid H\cdot K]}{P[Rc \mid \sim H\cdot K]} > (p+(1-p)/r) \times \frac{P[\sim Bc \mid H\cdot K]}{P[\sim Bc \mid \sim H\cdot K]} \; .$$

This theorem presupposes nothing about the sizes of p, q, and r except what's already implied by Non-Triviality. In particular, it does not presuppose that P[~Bc | K] > P[Rc | K], or anything like that. The theorem begins by summarizing facts about the factors p, q, and r. It adds the fact that q > 1–p, which is derived from the Non-Triviality Assumptions. It then tells us how p, q, and r figure into the various *likelihood ratios* and various *ratios of likelihood ratios*.

First we see that the *standard* likelihood ratio that a given Raven will turn out to be Black is just 1/p, which has to be greater than 1. So this must constitute positive evidence for "All ravens are black". The theorem also shows that the *standard* likelihood ratio that a given non-Black object will turn out to be a non-Raven is 1 / (1 – (1–p)/q). This factor must also be greater than 1, so this must also constitute positive evidence for the Ravens hypothesis, H.[3]

The theorem next provides formulas for the *ratio of standard likelihood ratios*. It shows that with regard to the *standard* likelihood ratios, a Raven found to be Black is *more support* / *equal support* / *less support* for H than a non-Black object found to be a non-Raven *just when* q is *greater than* 1 / *equal to* 1 / *less than* 1, respectively. To see this clearly, first look at the formula (1 – (1–p)/q) / p. It shows that when q is quite large (which would be a very reasonable supposition), the ratio of *standard* likelihood ratios is only a tiny bit below 1/p > 1 (since the numerator (1 – (1–p)/q) will be just a smidgen below 1). The alternative version of the formula, (1–(1–p)/q) / (1–(1–p)), makes it clear that if q is near 1 but larger than 1, then the whole formula remains larger than 1 (since the numerator will be just a bit larger than the denominator, because a tiny bit less is subtracted from 1 in the numerator than in the denominator); so for q > 1, the *standard* likelihood ratios will continue to provide more evidential support from a Raven found to be Black than from a non-Black object found to be a non-Raven. This same formula shows that for q = 1 the ratio of standard likelihood ratios equals 1; so the evidential support from a Raven found to be Black is precisely the same as the evidential support from a non-Black object found to be a non-Raven. When q is just a bit below 1, the formula becomes less than 1 (since the numerator will be just a bit smaller than the denominator, because a tiny bit more is subtracted from 1 in the numerator than in the denominator); so for q < 1, a non-Black object found to be a non-Raven will supply a bit more support than a Raven found to be Black. Both versions of the formula go on to show that as q approaches its lower bound (1–p), the numerator approaches 0; so a Raven found to be Black supplies only an extremely tiny fraction of the support given by a non-Black object found to be a non-Raven.

Next the theorem acknowledges that from the definitions of q and r alone we get

$$q/r = (P[Rc \mid H{\cdot}K]/P[Rc \mid {\sim}H{\cdot}K]) / (P[{\sim}Bc \mid H{\cdot}K]/P[{\sim}Bc \mid {\sim}H{\cdot}K]),$$

which is the *ratio of the non-standard likelihood ratios*.

Finally, the theorem provides a formula for the *ratio of the full likelihood ratios*, which is given by the product of the *ratio of standard likelihood ratios* and the *ratio of the non-standard likelihood ratios*. This formula is really the key to resolving the Ravens issue. The bi-conditional

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[3] Because q > (1–p) > 0, so 1 > (1–p)/q > 0, so 1 > (1 – (1–p)/q) > 0, so 1 < 1/ (1 – (1–p)/q).

at the end of the theorem expresses the most essential implications for the resolution of the Ravens issue. It says that a Black Raven *incrementally supports* “All ravens are black” *more strongly* than does a non-Black non-Raven (according to the likelihood ratio measure) *just in case* the two non-standard likelihood ratios are related by the inequality:

$$P[Rc \mid H\cdot K]/P[Rc \mid {\sim}H\cdot K] > (p + (1-p)/r) \times (P[{\sim}Bc \mid H\cdot K]/P[{\sim}Bc \mid {\sim}H\cdot K]).$$

Thus, a Black Raven *incrementally supports* H *more strongly* than does a non-Black non-Raven *just when*

> *the degree of incremental support* for (or against) H due to an object that’s merely establish to be a Raven (as measured by the likelihood ratio) *must be larger than a multiplicative factor* (p + (1–p)/r) of *the incremental support* for (or against) H due to an object that’s merely establish to be non-Black.

Up to this point we have drawn on the Non-Triviality Assumption, *and nothing more*. However, in the context of the Ravens issue the results we’ve been investigating will primarily of interest in the case where $P[{\sim}Bc \mid H\cdot K] \geq P[Rc \mid H\cdot K]$ (i.e. where $r \geq 1$). Taking the confirmational probability $P[{\sim}Bc \mid H\cdot K]$ to be at least as large as $P[Rc \mid H\cdot K]$ would be extremely plausible supposition. For instance, if we think of c as a randomly selected medium sized object, $P[{\sim}Bc \mid H\cdot K]$ should be *much larger* than $P[Rc \mid H\cdot K]$, given typical background knowledge that the number of non-Black objects *far exceeds* the number of Ravens. In any case, supposing merely that $P[{\sim}Bc \mid H\cdot K] \geq P[Rc \mid H\cdot K]$, the multiplicative factor $(p + (1-p)/r)$ has to be less than or equal to 1 (since p is strictly between 0 and 1). In that case the following corollary goes right to the heart of the matter.

Corollary to the Ravens Theorem: Suppose Non-Trivality. And suppose in addition that $P[{\sim}Bc \mid H\cdot K] \geq P[Rc \mid H\cdot K]$ (i.e. $r \geq 1$). Then $1 \geq (p + (1-p)/r) > p > 0$.
Furthermore (given $r \geq 1$),

$$\frac{P[Bc\cdot Rc \mid H\cdot K]}{P[Bc\cdot Rc \mid {\sim}H\cdot K]} > \frac{P[{\sim}Bc\cdot{\sim}Rc \mid H\cdot K]}{P[{\sim}Bc\cdot{\sim}Rc \mid {\sim}H\cdot K]} \quad \textit{if and only if} \quad \text{either}$$

(1) $$\frac{P[Rc \mid H\cdot K]}{P[Rc \mid {\sim}H\cdot K]} > \frac{P[{\sim}Bc \mid H\cdot K]}{P[{\sim}Bc \mid {\sim}H\cdot K]} \quad \text{or}$$

(2) $$\frac{P[{\sim}Bc \mid H\cdot K]}{P[{\sim}Bc \mid {\sim}H\cdot K]} \geq \frac{P[Rc \mid H\cdot K]}{P[Rc \mid {\sim}H\cdot K]} > (p + (1-p)/r) \times \frac{P[{\sim}Bc \mid H\cdot K]}{P[{\sim}Bc \mid {\sim}H\cdot K]} \text{ and } r > 1\,;$$

If the mere fact that object c is non-Black, all on its own, could supply *much stronger incremental support* for H than the mere fact that object c is a Raven, then a non-Black non-Raven might turn out to supply *stronger support* for H than a Black Raven. But, if a mere non-

Black object is at best only a little more evidence for H than is a mere Raven, then a Black Raven must *incrementally support* H *more strongly* than a non-Black non-Raven. How much more strongly can a mere non-Black object incrementally support H than does a mere Raven while still permitting a Black Raven to incrementally support H more strongly than a non-Black non-Raven? The corollary answers this question precisely. It says that when $P[\sim Bc \mid H{\cdot}K] \geq P[Rc \mid H{\cdot}K]$ (i.e. when $r \geq 1$), the following disjunction provides a necessary and sufficient condition for a Black Raven to *support* H *more strongly* than does a non-Black non-Raven:

either (1) the incremental support for H conferred by a mere Raven is larger than the incremental support for H conferred by a mere non-Black object – i.e.

$$P[Rc \mid H{\cdot}K]/P[Rc \mid \sim H{\cdot}K] \;>\; P[\sim Bc \mid H{\cdot}K]/P[\sim Bc \mid \sim H{\cdot}K],$$

or else (2) although the incremental support for H conferred by a mere non-Black object may be larger than (or as large as) the incremental support for H conferred by a mere Raven, nevertheless, at least some fraction $(p + (1-p)/r)$ of the *degree of incremental support* for H conferred by a mere non-Black object (as measured by the likelihood ratio measure), must remain smaller than the *degree of incremental support* for H conferred by a mere Raven (as measured by the likelihood ratio measure) – i.e.

$$P[\sim Bc \mid H{\cdot}K]/P[\sim Bc \mid \sim H{\cdot}K] \;\geq\; P[Rc \mid H{\cdot}K]/P[Rc \mid \sim H{\cdot}K] \;>\; (p + (1-p)/r) \times P[\sim Bc \mid H{\cdot}K]/P[\sim Bc \mid \sim H{\cdot}K].$$

Finally, notice that if in some context the $r \geq 1$ supposition doesn't seem appropriate, we don't really *have to* rely on it for a resolution of the Ravens issue. The main theorem itself tells us all that's really required. For, given only the Non-Triviality Assumptions, it's both sufficient and necessary that for a Black Raven to support H more strongly than a non-Black non-Raven, the non-standard likelihood ratios are related by the inequality

$$P[Rc \mid H{\cdot}K]/P[Rc \mid \sim H{\cdot}K] \;>\; (p + (1-p)/r) \times (P[\sim Bc \mid H{\cdot}K]/P[\sim Bc \mid \sim H{\cdot}K]),$$

whatever the numerical value of r may be!

One final bit of housekeeping:

The equation for *the ratio of full likelihood ratios* in the Ravens Theorem also appears to suggest the following result:

$$(\dagger) \qquad \frac{P[Bc{\cdot}Rc \mid H{\cdot}K]}{P[Bc{\cdot}Rc \mid \sim H{\cdot}K]} \;>\; \frac{P[\sim Bc{\cdot}\sim Rc \mid H{\cdot}K]}{P[\sim Bc{\cdot}\sim Rc \mid \sim H{\cdot}K]} \quad \textit{if and only if}$$

$$\frac{P[Rc \mid H{\cdot}K]}{P[Rc \mid \sim H{\cdot}K]} \;>\; \frac{p}{1-(1-p)/q} \times \frac{P[\sim Bc \mid H{\cdot}K]}{P[\sim Bc \mid \sim H{\cdot}K]}\,.$$

So why does our resolution of the Ravens issue draw on a different bi-conditional,

$$(*)\qquad \frac{P[Bc\cdot Rc \mid H\cdot K]}{P[Bc\cdot Rc \mid {\sim}H\cdot K]} > \frac{P[{\sim}Bc\cdot{\sim}Rc \mid H\cdot K]}{P[{\sim}Bc\cdot{\sim}Rc \mid {\sim}H\cdot K]} \quad \textit{if and only if}$$

$$\frac{P[Rc \mid H\cdot K]}{P[Rc \mid {\sim}H\cdot K]} > (p + (1-p)/r) \times \frac{P[{\sim}Bc \mid H\cdot K]}{P[{\sim}Bc \mid {\sim}H\cdot K]}$$

instead of drawing on (†)?

The alternative inequality (†) does indeed hold. However, it turns out that a Black Raven is more incrementally confirming than a non-Black non-Raven *just when* the factor (p + (1–p)/r) is larger than the factor p/(1–(1–p)/q). So equation (*) is more telling than equation (†). That is, it turns out that for a Black Raven to be more confirming that a non-Black non-Raven we must have the following (necessary and sufficient) relationship among the non-standard likelihood ratios:[4]

$$(P[Rc \mid H\cdot K]/P[Rc \mid {\sim}H\cdot K]) > [p + (1-p)/r] \times (P[{\sim}Bc \mid H\cdot K]/P[{\sim}Bc \mid {\sim}H\cdot K])$$

$$> [p/(1 - (1-p)/q)] \times (P[{\sim}Bc \mid H\cdot K]/P[{\sim}Bc \mid {\sim}H\cdot K])$$

$$> p \times (P[{\sim}Bc \mid H\cdot K]/P[{\sim}Bc \mid {\sim}H\cdot K]) .$$

All of this is established by the version of the main theorem we provide in the Appendix.

Appendix

It's easy to check that Non-Triviality yields 1 > p = P[Bc | Rc·~H·K] > 0, and P[Rc | H·K] > 0, and P[~Bc | H·K]/P[Rc | H·K] = r > 0, and P[~Bc | ~H·K] > 0, and P[Rc | ~H·K] > 0. Define q = P[~Bc | ~H·K]/P[Rc | ~H·K]; and notice that q > 0 as well.

Ravens Theorem. Non-Trivality together with the definitions of factors p, q, and r suffice for:

$$r = P[{\sim}Bc \mid H\cdot K]/P[Rc \mid H\cdot K] > 0; \quad q = P[{\sim}Bc \mid {\sim}H\cdot K]/P[Rc \mid {\sim}H\cdot K] > 0;$$

$$1 > p = P[Bc \mid Rc\cdot{\sim}H\cdot K] > 0; \quad q > (1-p) > 0;$$

$$\frac{P[Bc \mid Rc\cdot H\cdot K]}{P[Bc \mid Rc\cdot{\sim}H\cdot K]} = 1/p > 1 \; ; \quad \frac{P[{\sim}Rc \mid {\sim}Bc\cdot H\cdot K]}{P[{\sim}Rc \mid {\sim}Bc\cdot{\sim}H\cdot K]} = 1/(1 - (1-p)/q) > 1 \; ;$$

[4] The necessity and sufficiency of this relationship doesn't depend at all on whether or not r ≥ 1.

$$\frac{P[Bc \mid Rc \cdot H \cdot K]}{P[Bc \mid Rc \cdot {\sim}H \cdot K]} \Big/ \frac{P[{\sim}Rc \mid {\sim}Bc \cdot H \cdot K]}{P[{\sim}Rc \mid {\sim}Bc \cdot {\sim}H \cdot K]} = \frac{1-(1-p)/q}{p} = \frac{1-(1-p)/q}{1-(1-p)} \;;$$

$$> 1 \text{ when } q > 1,$$
$$= 1 \text{ when } q = 1,$$
$$< 1 \text{ when } q < 1 \;\; \{\text{note: } q > (1-p)\};$$

$$\frac{P[Rc \mid H \cdot K]}{P[Rc \mid {\sim}H \cdot K]} \Big/ \frac{P[{\sim}Bc \mid H \cdot K]}{P[{\sim}Bc \mid {\sim}H \cdot K]} = q/r \;;$$

$$\frac{P[Bc \cdot Rc \mid H \cdot K]}{P[Bc \cdot Rc \mid {\sim}H \cdot K]} \Big/ \frac{P[{\sim}Bc \cdot {\sim}Rc \mid H \cdot K]}{P[{\sim}Bc \cdot {\sim}Rc \mid {\sim}H \cdot K]}$$

$$= \frac{1-(1-p)/q}{p} \times \frac{P[Rc \mid H \cdot K]}{P[Rc \mid {\sim}H \cdot K]} \Big/ \frac{P[{\sim}Bc \mid H \cdot K]}{P[{\sim}Bc \mid {\sim}H \cdot K]}$$

$$= [q-(1-p)] / (r \times p) \;;$$

$$\frac{P[Bc \cdot Rc \mid H \cdot K]}{P[Bc \cdot Rc \mid {\sim}H \cdot K]} > \frac{P[{\sim}Bc \cdot {\sim}Rc \mid H \cdot K]}{P[{\sim}Bc \cdot {\sim}Rc \mid {\sim}H \cdot K]} \quad \textit{if and only if}$$

$$\frac{P[Rc \mid H \cdot K]}{P[Rc \mid {\sim}H \cdot K]} > (p + (1-p)/r) \times \frac{P[{\sim}Bc \mid H \cdot K]}{P[{\sim}Bc \mid {\sim}H \cdot K]}$$

if and only if $(p + (1-p)/r) > p / (1 - (1-p)/q)$

if and only if $q > r \times p + (1-p)$.

Notice that both $(p + (1-p)/r) > p$ and $p/(1 - (1-p)/q) > p$.
Also notice that when $q > r \times p + (1-p)$: if $r \geq 1$, then $q > r \times p + (1-p) \geq 1$;
if $r \leq 1$, then $q/r > p + (1-p)/r \geq 1$.

proof: The first three equations come directly from the definitions of r, q, and p.

The fifth equation, $P[Bc \mid Rc \cdot H \cdot K]/P[Bc \mid Rc \cdot {\sim}H \cdot K] = 1/p > 1$, holds because $P[Bc \mid Rc \cdot H \cdot K] = 1$ and $1 > p > 0$.

The fourth and sixth equations follow from the following considerations:

$$P[\sim Rc \mid \sim Bc\cdot\sim H\cdot K] = \{1 - (P[Rc\cdot\sim Bc \mid \sim H\cdot K]/P[\sim Bc \mid \sim H\cdot K])\}$$

$$= \{1 - (P[\sim Bc \mid Rc\cdot\sim H\cdot K]\, P[Rc \mid \sim H\cdot K]/P[\sim Bc \mid \sim H\cdot K])\}$$

$$= \{1 - (1-p)/q\} > 0, \text{ since } P[\sim Rc \mid \sim Bc\cdot\sim H\cdot K] > 0; \text{ so } q > (1-p) > 0.$$

$1 > P[\sim Rc \mid \sim Bc\cdot\sim H\cdot K] > 0$ because:
give that $\sim H$ is logically equivalent to $\exists x(Rx\cdot\sim Bx)$, Non-Triviality provides $P[(\sim Bc\cdot Rc)\cdot\sim H \mid K] = P[(\sim Bc\cdot Rc) \mid K] > 0$, and $P[(\sim Bc\cdot\sim Rc)\cdot\sim H \mid K] > 0$, so $P[\sim Bc\cdot\sim H \mid K] = P[(\sim Bc\cdot Rc)\cdot\sim H \mid K] + P[(\sim Bc\cdot\sim Rc)\cdot\sim H \mid K] > P[(\sim Bc\cdot\sim Rc)\cdot\sim H \mid K] > 0$, so $1 > P[(\sim Bc\cdot\sim Rc)\cdot\sim H \mid K]/P[\sim Bc\cdot\sim H \mid K] = P[\sim Rc \mid \sim Bc\cdot\sim H\cdot K] > 0$.

Also, $1/\{1 - (1-p)/q\} > 1$, since: from Non-Triviality, $p < 1$; and $p < 1$ iff $0 < (1-p)$ iff $0 > -(1-p)/q$ iff $1 > (1 - (1-p)/q)$ iff $1/\{1 - (1-p)/q\} > 1$ (since $q > (1-p) > 0$).

Thus, $P[\sim Rc \mid \sim Bc\cdot H\cdot K]/P[\sim Rc \mid \sim Bc\cdot\sim H\cdot K] = 1/\{1 - (1-p)/q\} > 1$.

The seventh equation comes from the values of the *standard* likelihood ratios in the fifth and sixth equations:

$$(P[Bc \mid Rc\cdot H\cdot K]/P[Bc \mid Rc\cdot\sim H\cdot K]) / (P[\sim Rc \mid \sim Bc\cdot H\cdot K]/P[\sim Rc \mid \sim Bc\cdot\sim H\cdot K])$$

$$= (1/p)/(1/\{1 - (1-p)/q\}) = \{1-(1-p)/q\}/p = \{1-(1-p)/q\}/\{1-(1-p)\}.$$

We get the eighth equation from the definitions of q and r alone:

$$q/r = (P[\sim Bc \mid \sim H\cdot K]/P[Rc \mid \sim H\cdot K]) / (P[\sim Bc \mid H\cdot K]/P[Rc \mid H\cdot K])$$
$$= (P[Rc \mid H\cdot K]/P[Rc \mid \sim H\cdot K]) / (P[\sim Bc \mid H\cdot K]/P[\sim Bc \mid \sim H\cdot K]),$$

which is the ratio of the *non-standard* likelihood ratios.

The ninth equation uses the seventh and eighth equations to get the *ratio of the full likelihood ratios* from the product of the *ratio of standard likelihood ratios* and the *ratio of the non-standard likelihood ratios*:

$$\frac{P[Bc\cdot Rc \mid H\cdot K]}{P[Bc\cdot Rc \mid \sim H\cdot K]} \Big/ \frac{P[\sim Bc\cdot\sim Rc \mid H\cdot K]}{P[\sim Bc\cdot\sim Rc \mid \sim H\cdot K]}$$

$$= \frac{1 - (1-p)/q}{p} \times \frac{P[Rc \mid H\cdot K]}{P[Rc \mid \sim H\cdot K]} \Big/ \frac{P[\sim Bc \mid H\cdot K]}{P[\sim Bc \mid \sim H\cdot K]}$$

$$= ([1 - (1-p)/q]/p) \times (q/r) = [q - (1-p)] / (r\times p).$$

The tenth “equation” is a series of bi-conditionals. The first bi-conditional follows from the following considerations:

Notice that $[1 - (1-p)/q] > 0$ (since $q > (1-p)$, so $1 > (1-p)/q$, so $1 - (1-p)/q > 0$).

Now, from the ninth equation we have that

$$\frac{P[Bc\cdot Rc \mid H\cdot K]}{P[Bc\cdot Rc \mid \sim H\cdot K]} \Big/ \frac{P[\sim Bc\cdot \sim Rc \mid H\cdot K]}{P[\sim Bc\cdot \sim Rc \mid \sim H\cdot K]} > 1 \text{ iff}$$

$$\frac{1 - (1-p)/q}{p} \times \frac{P[Rc \mid H\cdot K]}{P[Rc \mid \sim H\cdot K]} \Big/ \frac{P[\sim Bc \mid H\cdot K]}{P[\sim Bc \mid \sim H\cdot K]} > 1 \text{ iff}$$

$$(P[Rc \mid H\cdot K]/P[Rc \mid \sim H\cdot K]) / (P[\sim Bc \mid H\cdot K]/P[\sim Bc \mid \sim H\cdot K]) > p/\{1 - (1-p)/q\}.$$

However, it turns out that this is not the most telling condition for the ratio of *full likelihood ratios* to be greater than 1 (which is the reason that the theorem in the main text doesn't use this relationship). Rather, from the ninth equation we also have the following result:

$$\frac{P[Bc\cdot Rc \mid H\cdot K]}{P[Bc\cdot Rc \mid \sim H\cdot K]} \Big/ \frac{P[\sim Bc\cdot \sim Rc \mid H\cdot K]}{P[\sim Bc\cdot \sim Rc \mid \sim H\cdot K]} > 1 \text{ iff } \quad [q - (1-p)] / (r\times p) > 1 \text{ iff}$$

$$[(q/r) - (1-p)/r] > p \text{ iff } (q/r) > p + (1-p)/r \text{ iff}$$

$$(P[Rc \mid H\cdot K]/P[Rc \mid \sim H\cdot K]) / (P[\sim Bc \mid H\cdot K]/P[\sim Bc \mid \sim H\cdot K]) > (p + (1-p)/r),$$

since $(P[Rc \mid H\cdot K]/P[Rc \mid \sim H\cdot K]) / (P[\sim Bc \mid H\cdot K]/P[\sim Bc \mid \sim H\cdot K]) = (q/r)$.

This yields the first bi-conditional of the tenth equation.

Thus, the above formula for the *ratio of full likelihood ratios* implies *both*

(1) $(P[Bc\cdot Rc \mid H\cdot K]/P[Bc\cdot Rc \mid \sim H\cdot K]) / (P[\sim Bc\cdot \sim Rc \mid H\cdot K]/P[\sim Bc\cdot \sim Rc \mid \sim H\cdot K]) > 1$ iff
$(P[Rc \mid H\cdot K]/P[Rc \mid \sim H\cdot K])/(P[\sim Bc \mid H\cdot K]/P[\sim Bc \mid \sim H\cdot K]) > p/[1 - (1-p)/q]$,

and

(2) $(P[Bc\cdot Rc \mid H\cdot K]/P[Bc\cdot Rc \mid \sim H\cdot K]) / (P[\sim Bc\cdot \sim Rc \mid H\cdot K]/P[\sim Bc\cdot \sim Rc \mid \sim H\cdot K]) > 1$ iff
$(P[Rc \mid H\cdot K]/P[Rc \mid \sim H\cdot K])/(P[\sim Bc \mid H\cdot K]/P[\sim Bc \mid \sim H\cdot K]) > (p + (1-p)/r)$.

However, (2) is more telling because we can also show that
$(P[Bc\cdot Rc \mid H\cdot K]/P[Bc\cdot Rc \mid \sim H\cdot K]) / (P[\sim Bc\cdot \sim Rc \mid H\cdot K]/P[\sim Bc\cdot \sim Rc \mid \sim H\cdot K]) > 1$ iff
$[p + (1-p)/r] > p/[1 - (1-p)/q]$ (we'll establish this in a moment).
To see that (2) is more telling, suppose that the *ratio of full likelihood ratios* is greater than 1, and suppose (as I'm claiming for now) that under this condition we must have

$[p + (1-p)/r] > p/[1 - (1-p)/q]$. Then both (1) and (2) hold, but (2) requires that $(P[Rc \mid H{\cdot}K]/P[Rc \mid {\sim}H{\cdot}K])$ *must be* a larger (fractional) multiple of $(P[{\sim}Bc \mid H{\cdot}K]/P[{\sim}Bc \mid {\sim}H{\cdot}K])$ than required by (1). And the larger (fractional) multiple turns out to be more important for the Corollary (below), because it places tighter bounds on how much smaller than $(P[{\sim}Bc \mid H{\cdot}K]/P[{\sim}Bc \mid {\sim}H{\cdot}K])$ the ratio $(P[Rc \mid H{\cdot}K]/P[Rc \mid {\sim}H{\cdot}K])$ is permitted to be while having Black Ravens remain more confirming than Non-black Non-ravens.

Here's proof of that

$(P[Bc{\cdot}Rc \mid H{\cdot}K]/P[Bc{\cdot}Rc \mid {\sim}H{\cdot}K]) / (P[{\sim}Bc{\cdot}{\sim}Rc \mid H{\cdot}K]/P[{\sim}Bc{\cdot}{\sim}Rc \mid {\sim}H{\cdot}K]) > 1$ iff

$[p + (1-p)/r] > p/[1 - (1-p)/q]$

(which, when combined with the first bi-conditional, yields the second bi-conditional of the tenth equation).

From the ninth equation we have that

$(P[Bc{\cdot}Rc \mid H{\cdot}K]/P[Bc{\cdot}Rc \mid {\sim}H{\cdot}K]) / (P[{\sim}Bc{\cdot}{\sim}Rc \mid H{\cdot}K]/P[{\sim}Bc{\cdot}{\sim}Rc \mid {\sim}H{\cdot}K]) > 1$ iff
$(q - (1-p)) / (r \times p) > 1$ iff $([q/r] - (1-p)/r) > p$ iff $[q/r] - [p + (1-p)/r] > 0$ iff
$[1/r] - [1/q][p + (1-p)/r] > 0$ iff $[(1-p)/r] - [(1-p)/q]\,[p + (1-p)/r] > 0$ iff
$[p + (1-p)/r] - [(1-p)/q]\,[p + (1-p)/r] > p$ iff $[1 - (1-p)/q]\,[p + (1-p)/r] > p$ iff
$[p + (1-p)/r] > p/[1 - (1-p)/q]$, since $[1 - (1-p)/q] > 0$
(because $q > (1-p)$, so $1 > (1-p)/q$, so $1-(1-p)/q) > 0$).

To prove the third bi-conditional of the tenth equation, follow the chain of "iff"s in the previous derivation back up the chain from
$[p + (1-p)/r] > p/[1 - (1-p)/q]$ to $(q - (1-p)) / (r \times p) > 1$, yielding:
$[p + (1-p)/r] > p/[1 - (1-p)/q]$ iff $(q - (1-p)) / (r \times p) > 1$ iff
$(q - (1-p)) > r \times p$ iff $q > r \times p + (1-p)$.

Notice that when $q > r \times p + (1-p)$: if $r \geq 1$ we have $q > r \times p + (1-p) \geq 1$; and when $r \leq 1$, $1/r \geq 1$, so $(1-p)/r \geq (1-p)$, so $q/r > p + (1-p)/r \geq p + (1-p) = 1$, so $q > r$.

Corollary to the Ravens Theorem: Suppose Non-Trivality. And suppose in addition that $P[{\sim}Bc \mid H{\cdot}K] \geq P[Rc \mid H{\cdot}K]$ (i.e. $r \geq 1$). Then $1 \geq (p + (1-p)/r) > p > 0$.
Furthermore (given $r \geq 1$),

$$\frac{P[Bc{\cdot}Rc \mid H{\cdot}K]}{P[Bc{\cdot}Rc \mid {\sim}H{\cdot}K]} > \frac{P[{\sim}Bc{\cdot}{\sim}Rc \mid H{\cdot}K]}{P[{\sim}Bc{\cdot}{\sim}Rc \mid {\sim}H{\cdot}K]} \quad \textit{if and only if} \quad \text{either}$$

$$(1) \qquad \frac{P[Rc \mid H{\cdot}K]}{P[Rc \mid {\sim}H{\cdot}K]} > \frac{P[{\sim}Bc \mid H{\cdot}K]}{P[{\sim}Bc \mid {\sim}H{\cdot}K]} \qquad \text{or}$$

$$(2) \qquad \frac{P[{\sim}Bc \mid H{\cdot}K]}{P[{\sim}Bc \mid {\sim}H{\cdot}K]} \geq \frac{P[Rc \mid H{\cdot}K]}{P[Rc \mid {\sim}H{\cdot}K]} > (p + (1-p)/r) \times \frac{P[{\sim}Bc \mid H{\cdot}K]}{P[{\sim}Bc \mid {\sim}H{\cdot}K]} \text{ and } r > 1.$$

proof: Suppose throughout that $r \geq 1$.

We already have $1 > p > 0$, so $1 > 1-p > 0$. Then, for $r > 1$, we must have $(1-p) > (1-p)/r > 0$; thus $1 = p + (1-p) \geq p + (1-p)/r > p$.

The theorem already gave us:

$$(P[Bc \cdot Rc \mid H \cdot K]/P[Bc \cdot Rc \mid {\sim}H \cdot K]) > (P[{\sim}Bc \cdot {\sim}Rc \mid H \cdot K]/P[{\sim}Bc \cdot {\sim}Rc \mid {\sim}H \cdot K]) \quad \text{iff}$$
$$P[Rc \mid H \cdot K]/P[Rc \mid {\sim}H \cdot K] > (p + (1-p)/r) \times (P[{\sim}Bc \mid H \cdot K]/P[{\sim}Bc \mid {\sim}H \cdot K])$$

and we just proved that $1 \geq (p + (1-p)/r) > 0$ when $r \geq 1$, so

$$(P[{\sim}Bc \mid H \cdot K]/P[{\sim}Bc \mid {\sim}H \cdot K]) \geq (p + (1-p)/r) \times (P[{\sim}Bc \mid H \cdot K]/P[{\sim}Bc \mid {\sim}H \cdot K]).$$

Thus, $P[Bc \cdot Rc \mid H \cdot K]/P[Bc \cdot Rc \mid {\sim}H \cdot K] > P[{\sim}Bc \cdot {\sim}Rc \mid H \cdot K]/P[{\sim}Bc \cdot {\sim}Rc \mid {\sim}H \cdot K]$ iff

either $(P[Rc \mid H \cdot K]/P[Rc \mid {\sim}H \cdot K]) > (P[{\sim}Bc \mid H \cdot K]/P[{\sim}Bc \mid {\sim}H \cdot K])$

or $(P[{\sim}Bc \mid H \cdot K]/P[{\sim}Bc \mid {\sim}H \cdot K]) \geq (P[Rc \mid H \cdot K]/P[Rc \mid {\sim}H \cdot K]) >$
$(p + (1-p)/r) \times (P[{\sim}Bc \mid H \cdot K]/P[{\sim}Bc \mid {\sim}H \cdot K])$

{where $r > 1$ whenever the second disjunct holds, because it requires that
$(P[{\sim}Bc \mid H \cdot K]/P[{\sim}Bc \mid {\sim}H \cdot K]) >$
$(p + (1-p)/r) \times (P[{\sim}Bc \mid H \cdot K]/P[{\sim}Bc \mid {\sim}H \cdot K])$}.